WELL-POSEDNESS FOR A CAUCHY FRACTIONAL DIFFERENTIAL PROBLEM WITH HILFER TYPE FRACTIONAL DERIVATIVE

BY

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الإهـــداء

أهدي هذا العمل

إلى والــــدي العـــزيزيــن ..... ثمرة من ثمار تربيتهم وغرســـــــــهم

إلى زوجتى العزيزه و اولادي.... وفاءً لهم ولصبرهم وتضحيتهـــــــــم

إلى أخـــــــــواتـــــــــى .... رداً لبعض معروفهن وإحسانهـــــــــن

DEDICATED

TO

MY PARENTS AND FAMILY

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I am most grateful to Almighty ALLAH, the Beneficent, the Merciful, for enabling me to complete this work. Peace and blessings of ALLAH be upon his Last messenger Mohammed (Sallallah-Alaihe-Wasallam) and his family, who guided us to the right path.

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THESIS ABSTRACT

Name: **MOHAMMED DAHAN AHMED KASSIM**

Title: **WELL-POSEDNESS FOR A CAUCHY FRACTIONAL DIFFERENTIAL PROBLEM WITH HILFER TYPE FRACTIONAL DERIVATIVE**

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We considered the initial value problem for a class of nonlinear fractional differential equations that involve Hilfer and Hilfer-Hadamard fractional derivatives. The first one interpolates the Riemann-Liouville fractional derivative and the Caputo fractional derivative and the second one interpolates the Hadamard fractional derivative and its Caputo counterpart. We obtained existence and uniqueness results for both types in suitably selected underlying spaces. We proved stability results for a certain class of nonlinearity. Also, we established non-existence results in case the nonlinearity is of polynomial type using the test function method.

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ملخص الرسالة

الاســـــــــــم: محمد دحان أحمد قاسم

عنوان الرســـــالة: الصياغة الجيدة لمسألة كوشي لمعادلات تفاضلية ذات رتب غير صحيحة تحتوي على مشتقة هيلفر

الدرجـــــة العلمية: ماجستير

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درسنا مسالة القيمة الابتدائية لنوع من المعادلات التفاضلية غير الخطية ذات رتب غير صحيحة تحتوي على مشتقة هيلفر و هيلفر-هدمارد. النوع الاول يستكمل مشتقة ريمان-ليوفل ومشتقة كابوتو و النوع الثاني يستكمل مشتقة هدمارد ومشتقة كابوتو المناظرة لها. حصلنا على نتائج وجود و وحدانية الحل لكلا النوعين في فضاء تم اختياره بطريقة مناسبة. كما اثبتنا إستقرار الحل لنوع معين من الدوال غير الخطية. ايضا، في حالة كون الدالة الغير خطية من نوع كثيرة الحدود تم اثبات عدم وجود الحل باستخدام طريقة الدالة الاختبارية.

Chapter 1

INTRODUCTION

1.1 What is Fractional Calculus

Fractional calculus is the field of mathematical analysis which deals with the investigation and applications of integrals and derivatives of arbitrary order, unlike the classical calculus, the order of the integrals and derivatives are not necessarily of integer order. It could be any real or complex number or even a variable order.

1.2 First contributions

The idea of fractional derivative goes back to more than three hundred years. In fact, fractional calculus may be considered an old and yet novel topic. It is an old topic since, starting from some speculations of G.W. Leibniz (1697) and L. Euler (1730), it has been developed up to nowadays. A list of mathematicians, who have provided important contributions up to the middle of the last century, includes P. S. Laplace (1812), J. B. J. Fourier (1822), N. H. Abel (1823-1826), J. Liouville (1832-1873), H. Laurent (1884), P. A. Nekrassov (1888), A. Krug (1890), J. Hadamard (1892) , O. Heaviside (1892-1912), S. Pincherle (1902), G. H. Hardy and J.E. Littlewood (1917-1928), H. Weyl (1917), P. Lévy (1923), A. Marchaud (1927), H. T. Davis (1924-1936), A. Zygmund (1935-1945), E. R. Love (1938-1996), A. Erdélyi (1939-1965), H. Kober (1940), D. V. Widder (1941), M. Riesz (1949),...

1.3 First books

The first book entirely devoted to the subject is the book by Oldham and Spanier [79]. Other books are by: Samko, Kilbas and Marichev [89], Podlubny [83], Kilbas, Strivastava and Trujillo [55], Mainardi [68]. For the first monograph the merit is ascribed to K. B. Oldham and J. Spanier, see [79], who, after a joint collaboration started in 1968, published a book devoted to fractional calculus in 1974. Nowadays, the list of texts and proceedings devoted solely or partly to fractional calculus and its applications includes about a dozen of titles [87, 88], among which the encyclopedic treatise by Samko, Kilbas & Marichev [89] is the most prominent.

Furthermore, we draw the attention of the reader to the treatises by Davis [17], Erdélyi [36], Gelfand and Shilov [43], Dzherbashian [25, 26], Caputo [16], Babenko [8], Gorenflo and Vessella [45], which contain a detailed analysis of some mathematical aspects and/or physical applications of fractional calculus, although without explicit mention in their titles.

1.4 Journals

There are three journals in the subject: "Journal of Fractional Calculus", "Fractional Calculus and Applied Analysis" and the recent one "Fractional Dynamic Systems". These journals publish work exclusively on fractional calculus and analysis. Special issues are also being published in other journals (like : Advances in Difference equations) from time to time.

1.5 First conferenc

Fractional calculus may also be considered as a novel topic, since only from a little more than twenty years it has been the subject of specialized conferences and treatises. For the first conference the merit is ascribed to B. Ross who organized the First Conference on Fractional Calculus and its Applications at the University of New Haven in June 1974, and edited the proceedings, see [87].

1.6 Fractional Differential Equations

Fractional differential equations have been of great interest recently because of both the intensive development of the theory of fractional calculus itself and the applications of such constructions in various sciences such as physics, mechanics, chemistry, finance, engineering, possibly including fractal phenomena, etc. For details,the reader is referred to the monographs of Lakshmikantham et al. [62], Miller and Ross [73] and Samko et al. [89], Podlubny [83], Kilbas [55] and the papers of Diethelm et al. [21, 22, 23], Gaul et al [42], Glockle and Nonnenmacher [44], Mainardi [67], Metzler et al. [71], Momani and Hadid [75], Momani et al. [76], Podlubny et al. [84], Yu and Gao [99] and the references therein.

1.7 Physical Systems

It should be noted that there are a growing number of physical systems whose behavior can be compactly described using fractional-order system theory. Of specific interest to engineers are viscoelastic materials [12, 57, 58, 92], electrochemical processes [50, 94], long lines [48], dielectric polarization [93], colored noise [69], and chaos [46]. With the growing number of applications, it is important to establish a clear initialized system theory for these fractional-order systems, so that it may be accessible to the general engineering and scientific communities.

1.8 First experiments and interpretations

Four decades ago engineers were ignoring fractional derivatives and avoided dealing with them directly because of their complexity and sometimes inconsistency. There was no solid basis as their study is very different from the integer order case. The interest to fractional calculus has been accelerated in the past three decades after the publication of the three papers of Bagley and Torvik [9-11] and the paper by Podlubny [83]. The authors proved by experiments that when using fractional derivatives for viscoelastic materials leads to:

- experimental data (amplitude/frequency and dispersions) are in agreement within a broad frequency range,

- time domain responses as stress relaxation and creep are well represented,

- the number of parameters is significantly reduced.

In addition to that, Podlubny established in [85] a geometric interpretation of the fractional integral and a physical interpretation of the fractional derivative.

1.9 Different kinds of derivatives

Several kinds of fractional derivatives have been defined: The Riemann-Liouville, Weyl, Weyl-Liouville, Chen, Bessel, Marchaud, Grunwald-Letnikov, Dzerbashyan, Hadamard derivative, etc... The first two are the most used ones. The first one is commonly used especially by mathematicians but it does not work with the usual initial conditions as the physical meaning of initial fractional derivatives is not clear and there are no ways how to measure them. Solutions must behave like the inverse of a power type function nearly the initial point. Physicists and engineers preferred the Caputo derivative which is compatible with the usual initial conditions although we can find nowadays several tentative justifying the use of fractional initial conditions.

1.10 Usefulness of fractional calculus

Fractional calculus has attracted a large number of researchers and has become very popular in numerous fields of sciences and engineering. It has been shown that fractional derivatives and fractional integrals are very successful in describing for instance anomalous kinetics and continuous time random walks. In general it has been confirmed that fractional derivatives are more suitable and adequate than derivatives of integer orders for the description of properties of many materials (especially materials with memory) and hereditary phenomena and processes. There are many other complex systems in atmospheric diffusion of pollution, network traffic, anomalous diffusion, chaotic processes, biology, medicine, modeling and identification, electronics and wave propagation, mechanics, astrophysics, signal processing, chaotic dynamics, optics and porous media where fractional calculus has applications.

1.11 History of the problem

For the Riemann-Liouville Fractional Differential Problem (R-L FDP)

$$\begin{cases} \left(D_{a+}^{\alpha} y\right)(x) = f\left[x, y(x)\right], \ x > a, \quad 0 < \alpha < 1 \\ \left(D_{a+}^{\alpha-1} y\right)(a+) = c, \end{cases} \tag{1.1}$$

and the Caputo Fractional Differential Problem (C FDP)

$$\begin{cases} \left({}^{C}D_{a+}^{\alpha} y\right)(x) = f\left[x, y(x)\right], \ x > a, \quad 0 < \alpha < 1 \\ y(a+) = c, \end{cases} \tag{1.2}$$

we may quote almost all the papers in our references.

Most of these publications are concerned with the issue of existence and uniqueness. We can cite the first work of Pitcher and W. Sewell [82] who in 1938 studied the nonlinear fractional differential equation (1.1) when f is bounded and Lipschitz continuous with respect to y. But special cases of f have been treated before that by some researches such as O'Shaughnessy [80] (1918), Post [86] (1919) and Mandelbrojt

[70] (1925).

The R-L FDP (1.1) was studied for the first time by Al-Bassam [6] (1965) in the space of continuous functions when f is a real-valued continuous and Lipschitz in a certain domain.

Then we can find the papers of Al-Abedeen [4] and Al-Abedeen and Arora [5], Arora and Alshamani [7], Tazali [95], Tazali and Karim [96] when the nonlinearity satisfies a generalized Lipschitz condition using different methods. We also quote the works of Leskovskij [64], Semenchuk [90], Luszczki and Rzepecki [66], El-Sayed and El-Sayed and collaborators [27-35]. In almost all these publications there was a serious problem regarding the equivalence between the Fractional Differential Problem and its corresponding Integral Equation. These publications were followed by many others written by Delbosco and Rodino [18], Hayek et al. [47], Podlubny [83], Kilbas et al. [51,52], Diethelm and Ford [24], Kilbas and Marzan ([53] and [54] where the equivalence was proved rigorously, using successive expansion and fixed point theorems. In parallel we can find many works on special nonlinearities f, the linear case and also on fractional differential problems involving other types of fractional derivatives than the Riemann-Liouville fractional derivative.

More recently, in addition to the Picard-type iterative process and the approximate-iterative method by Dzjadyk, several kinds of fixed point theorems have been used in [2,14,3,53,98,65,99,97,54,59,61,13,24,78,91,100].

Some stability results have been established in [1], [34], [27-34], [56], [35], [37], [77].

Finally we mention that the quasilinearization method has been generalized to the fractional order case in [19,20].

1.12 Our contribution

In [49], Hilfer introduced a new type of fractional derivative which we name after him in this thesis

$$\left(D_{a\pm}^{\alpha,\beta}y\right)(x)=\pm\left(I_{a\pm}^{\beta(1-\alpha)}\frac{d}{dx}I_{a\pm}^{(1-\beta)(1-\alpha)}y\right)(x)\,. \tag{1.3}$$

This fractional derivative may be viewed as interpolating the Riemann-Liouville derivative and the Caputo derivative in the sense that for $\beta=0$ we recover the Riemann-Liouville fractional derivative (1.1) and for $\beta=1$ we find the Caputo fractional derivative (1.2).

We intend to study the basic fractional differential problem with the fractional derivative of Hilfer type :

$$\begin{cases} \left(D_{a+}^{\alpha,\beta}y\right)(x)=f\left[x,y\left(x\right)\right],\ x>a, \quad 0<\alpha<1, \quad 0\leq\beta\leq 1 \\ \left(I_{a+}^{(1-\beta)(1-\alpha)}y\right)(a)=c. \end{cases} \tag{1.4}$$

In this thesis we are interested in problem (1.4). It had appeared for the first time in [49] but we could not find any mathematical basic treatment (well-posedness, stability, asymptotic behavior, ...) for this problem.

The topics to be covered in this thesis are the following

- Existence and uniqueness of solutions
- Some stability issues
- Non-existence and/or blow up of solutions.

As it is mentioned earlier, the equation with Riemann-Liouville derivative or Caputo is supplemented by an initial value condition which depends on the considered type of fractional derivative. These two problems are mathematically and physically different. Nevertheless, we hope that our investigation will lead to some sort of unification of both treatments.

For the existence and uniqueness we will first establish an equivalence between the FDP (1.4) and an integral equation. This integral equation will determine the operator T for which we apply later a suitable fixed point theorem. We will follow the nice proof in the book of Kilbas et al. [55] where they first establish the equivalence between the Fractional Differential Problem and its corresponding Nonlinear Volterra Integral Equation, then using the contraction principal they establish the existence and uniqueness of a solution.

This equivalence is crucial and more important is the space where it holds true. This space is delicate and should be carefully selected as fractional derivatives (apart from Caputo fractional derivative) cause some regularity problems as they (and the functions) may be singular at the initial point.

For the stability we will exploit the "initial decay" of solutions, that is the behavior of solutions nearby the initial point and determine sufficient conditions on the nonlinearity which allow us to push and keep this behavior for all time in case of global existence. We apply some desingularization techniques to be able to use a singular version of the Gronwall inequality.

There are several ways to investigate the non-existence and the blow up of solutions. We use the test function method developed by Pohozaev and Mitidieri [74]. This technique is very efficient and it has been used successfully for the case of Riemann-

Liouville fractional derivative. It works perfectly for a polynomial source. We will follow in particular the paper by Laskri and Tatar [63].

This thesis is organized as follows:

In chapter two, we present the basic definitions, lemmas, properties and notation needed later in this thesis. In chapter three, the existence and uniqueness of a solution, a stability result and a non-existence result are stated and proved. The same kinds of questions are addressed for the corresponding Hilfer-Hadamard type problem in chapter four. Some recommendations for future work are presented in chapter five.

Chapter 2

PRELIMINARIES

In this chapter we present some definitions, lemmas, properties and notation which will be used in our theorems later.

2.1 Spaces of Integrable Functions and Weighted Continuous Functions

In this section we present the definition of p-integrable functions, weighted p-integrable functions and some weighted spaces of continuous functions. We also give some characterizations of these modified spaces which will be used later. Moreover, some important embeddings are stated.

Definition 2.1.1. [55, p. 1]: *Let $\Omega = (a,b)$ $(-\infty \leq a < b \leq \infty)$ be a finite or infinite interval of the real axis $\mathbb{R} = (-\infty, \infty)$. We denote by $L_p(a,b)$ $(1 \leq p \leq \infty)$ the set of those Lebesgue real-valued measurable functions f on $[a,b]$ for which $\|f\|_p < \infty$, where*

$$\|f\|_p = \left(\int_a^b |f(t)|^p \, dt \right)^{1/p}, \quad 1 \leq p < \infty,$$

and

$$\|f\|_\infty = \operatorname*{ess\,sup}_{a \leq x \leq b} |f(x)| .$$

Here $\operatorname{ess\,sup} |f(x)|$ is the essential supremum of the function $|f(x)|$.

Definition 2.1.2. [55, p. 1]: *The space $X_c^p(a,b)$ $(c \in \mathbb{R},\ 1 \leq p \leq \infty)$ consists of those real-valued Lebesgue measurable functions g on (a,b) for which $\|g\|_{X_c^p} < \infty$,* *where*

$$\|g\|_{X_c^p} = \left(\int_a^b |t^c g(t)|^p \frac{dt}{t}\right)^{1/p}, \quad 1 \le p < \infty, \ c \in \mathbb{R}$$

and

$$\|g\|_{X_c^\infty} = \text{ess} \sup_{a \le x \le b} |x^c g(x)|, \ c \in \mathbb{R}.$$

In particular, when $c = 1/p$ we see that $X_{1/p}^p(a,b) = L_p(a,b)$.

Definition 2.1.3. [55, p. 3]: *Let $\Omega = [a,b]$ $(-\infty < a < b < \infty)$ and $m \in \mathbb{N}_0 = \{0, 1, ...\}$. We denote by $C^m(\Omega)$ the space of functions f which are m times continuously differentiable on Ω with the norm*

$$\|f\|_{\mathbf{C}^m} = \sum_{k=0}^m \left\|f^{(k)}\right\|_{\mathbf{C}} = \sum_{k=0}^m \max_{x \in \Omega} \left|f^{(k)}(x)\right|, \quad m \in \mathbb{N}_0.$$

In particular, for $m = 0$, $C^0(\Omega) \equiv C(\Omega)$ is the space of continuous function f on Ω with the norm

$$\|f\|_{\mathbf{C}} = \max_{x \in \Omega} |f(x)|.$$

Definition 2.1.4. [55, p. 4]: *Let $\Omega = [a,b]$ be a finite interval and $0 \le \gamma < 1$, we introduce the weighted space $C_\gamma[a,b]$ of continuous functions f on $(a,b]$*

$$C_\gamma[a,b] = \{f : (a,b] \to \mathbb{R} \colon (x-a)^\gamma f(x) \in C[a,b]\}.$$

In the space $C_\gamma[a,b]$, we define the norm

$$\|f\|_{C_\gamma} = \|(x-a)^\gamma f(x)\|_C, \quad C_0[a,b] = C[a,b].$$

Definition 2.1.5. [55, p. 4]: *For $n \in \mathbb{N}$ we denote by $C_\gamma^n[a,b]$ $(0 \le \gamma < 1)$ the Banach space of functions f which are continuously differentiable on $[a,b]$ up to order $n-1$ and have the derivative $f^{(n)}$ of order n on $(a,b]$ such that $f^{(n)} \in C_\gamma[a,b]$*

$$C_\gamma^n[a,b] = \left\{ f \in C^{n-1}[a,b] : f^{(n)} \in C_\gamma[a,b] \right\},$$

with the norm

$$\|f\|_{C_\gamma^n} = \sum_{k=0}^{n-1} \left\| f^{(k)} \right\|_C + \left\| f^{(n)} \right\|_{C_\gamma}, \quad C_\gamma^0[a,b] = C_\gamma[a,b].$$

From this definition we have the following characterization of the space $C_\gamma^n[a,b]$.

Lemma 2.1.1. [55, p. 4]: *Let $n \in \mathbb{N} = \{1,2,...\}$ and $0 \le \gamma < 1$. The space $C_\gamma^n[a,b]$ consists of those and only those functions f which can be represented in the form*

$$f(x) = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} \varphi(t)\, dt + \sum\nolimits_{k=0}^{n-1} c_k (x-a)^k,$$

where $\varphi \in C_\gamma[a,b]$ and c_k $(k = 0,1,...,n-1)$ are arbitrary constants.

Moreover,

$$\varphi(t) = f^{(n)}(t), \qquad c_k = \frac{f^{(k)}(a)}{k!} \qquad (k = 0,1,...,n-1).$$

Definition 2.1.6. [55, p. 4]: *Let $\Omega = [a,b]$ $(0 < a < b < \infty)$ be a finite interval and $0 \le \gamma < 1$, we introduce the weighted space $C_{\gamma,\log}[a,b]$ of continuous functions g*

on $(a,b]$

$$C_{\gamma,\log}[a,b]=\left\{g:(a,b]\rightarrow\mathbb{R}\colon\left(\log\frac{x}{a}\right)^{\gamma}g(x)\in C[a,b]\right\}.$$

In the space $C_{\gamma,\log}[a,b]$, *we define the norm*

$$\|g\|_{C_{\gamma,\log}}=\left\|\left(\log\frac{x}{a}\right)^{\gamma}g(x)\right\|_{C},\quad C_{0,\log}[a,b]=C[a,b].$$

Definition 2.1.7. [55, p. 5]: *Let* $\delta=x\frac{d}{dx}$ be the δ-*derivative*, f*or* $n\in\mathbb{N}$ *we denote by* $C^{n}_{\delta,\gamma}[a,b]$ *(*$0\leq\gamma<1$*) the Banach space of functions* g *which have continuous* δ-*derivatives on* $[a,b]$ *up to order* $n-1$ *and have the derivative* $\delta^{n}g$ *of order* n *on* $(a,b]$ *such that* $\delta^{n}g\in C_{\gamma,\log}[a,b]$

$$C^{n}_{\delta,\gamma}[a,b]=\left\{\delta^{k}g\in C[a,b],\quad k=0,...,n-1,\quad\delta^{n}g\in C_{\gamma,\log}[a,b]\right\}$$

with the norm

$$\|g\|_{C^{n}_{\delta,\gamma}}=\sum_{k=0}^{n-1}\left\|\delta^{k}g\right\|_{C}+\|\delta^{n}g\|_{C_{\gamma,\log}}.$$

When $n=0$ *we set*

$$C^{0}_{\delta,\gamma}[a,b]=C_{\gamma,\log}[a,b].$$

From this definition we have the following characterization of the space $C^{n}_{\delta,\gamma}[a,b]$.

Lemma 2.1.2. [55, p. 5]: *Let* $0<a<b<\infty$, $n\in\mathbb{N}_{0}=\{0,1,...\}$ *and* $0\leq\gamma<1$. *The space* $C^{n}_{\delta,\gamma}[a,b]$ *consists of those and only those functions* g *which can be* *represented in the form*

$$g(x) = \frac{1}{(n-1)!} \int_a^x \left(\log \frac{x}{t}\right)^{n-1} \varphi(t) \frac{dt}{t} + \sum_{k=0}^{n-1} d_k \left(\log \frac{x}{a}\right)^k,$$

where $\varphi \in C_{\gamma,\log}[a,b]$ *and* d_k *(*$k = 0, 1, ..., n-1$*) are arbitrary constants.*

Moreover,

$$\varphi(t) = (\delta^n g)(t), \qquad d_k = \frac{(\delta^k g)(a)}{k!} \qquad (k = 0, 1, ..., n-1).$$

From the Definitions 2.1.3 to 2.1.7 we have the following embeddings.

Property 2.1.1. [55, p. 5]: *Let* $n \in \mathbb{N}_0 = \{0, 1, ...\}$ *and let* μ_1 *and* μ_2 *be real numbers such that*

$$0 \leq \mu_1 \leq \mu_2 < 1.$$

The following embedding hold:

$$C^n[a,b] \to C^n_{\mu_1}[a,b] \to C^n_{\mu_2}[a,b],$$

with

$$\|f\|_{C^n_{\mu_2}} \leq K \|f\|_{C^n_{\mu_1}}, \quad K = \min\left[1, (b-a)^{\mu_2-\mu_1}\right];$$

$$C^n_\delta[a,b] \to C^n_{\delta,\mu_1}[a,b] \to C^n_{\delta,\mu_2}[a,b],$$

with

$$\|f\|_{C^n_{\delta,\mu_2}} \leq K_\delta \|f\|_{C^n_{\delta,\mu_1}}, \quad K_\delta = \min\left[1, \left(\log \frac{b}{a}\right)^{\mu_2-\mu_1}\right].$$

In particular,

$$C[a,b] \to C_{\mu_1}[a,b] \to C_{\mu_2}[a,b],$$

with

$$\|f\|_{C_{\mu_2}} \leq (b-a)^{\mu_2-\mu_1} \|f\|_{C_{\mu_1}};$$

$$C[a,b] \to C_{\mu_1,\log}[a,b] \to C_{\mu_2,\log}[a,b],$$

with

$$\|f\|_{C_{\mu_2,\log}} \leq \left(\log \frac{b}{a}\right)^{\mu_2-\mu_1} \|f\|_{C_{\mu_1,\log}}.$$

Definition 2.1.8. [55, p. 24]: *The Euler gamma function $\Gamma(z)$ is defined by the so–called Euler integral of the second kind:*

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt, \; z > 0,$$

where $t^{z-1} = e^{(z-1)\log t}$.

2.2 Riemann-Liouville Fractional Integrals and Fractional Derivatives

In this section we introduce the definitions of the Riemann-Liouville fractional integrals and fractional derivatives on a finite interval of the real line and present some of their properties in the space of continuous functions.

Definition 2.2.1. [55, p. 69]: *Let $\Omega = [a,b]$ $(-\infty < a < b < \infty)$ be a finite* *interval on the real axis* $\mathbb{R}$. *The Riemann-Liouville left-sided fractional integral* $I^{\alpha}_{a+}f$ *of order* $\alpha > 0$ *is defined by*

$$\left(I^{\alpha}_{a+}f\right)(x) := \frac{1}{\Gamma(\alpha)}\int_a^x \frac{f(t)}{(x-t)^{1-\alpha}}dt, \ \ (a < x < b, \ \alpha > 0)$$

provided that the integral exists. Here $\Gamma(\alpha)$ *is the Gamma function. When* $\alpha = 0$, *we set*

$$I^{0}_{a+}f = f.$$

Definition 2.2.2. [55, p. 69]: *Let* $\Omega = [a, b]$ $(-\infty < a < b < \infty)$ *be a finite interval on the real axis* $\mathbb{R}$. *The Riemann-Liouville right-sided fractional integral* $I^{\alpha}_{b-}f$ *of order* $\alpha > 0$ *is defined by*

$$\left(I^{\alpha}_{b-}f\right)(x) := \frac{1}{\Gamma(\alpha)}\int_x^b \frac{f(t)}{(t-x)^{1-\alpha}}dt, \ \ (a \leq x < b, \ \alpha > 0)$$

provided that the integral exists. When $\alpha = 0$, *we define*

$$I^{0}_{b-}f = f.$$

Definition 2.2.3. [55, p. 70]: *Let* $\Omega = [a, b]$ $(-\infty < a < b < \infty)$ *be a finite interval on the real axis* $\mathbb{R}$. *The Riemann-Liouville left-sided fractional derivative* $D^{\alpha}_{a+}f$ *of order* α $(0 \leq \alpha < 1)$ *is defined by*

$$\left(D^{\alpha}_{a+}f\right)(x) = \frac{d}{dx}\left(I^{1-\alpha}_{a+}f\right)(x)$$

that is

$$\left(D_{a+}^{\alpha}f\right) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x \frac{f(t)}{(x-t)^{\alpha}}dt \ \left(x > a,\ 0 < \alpha < 1\right),$$

when $\alpha = 1$ we have $D_{a+}^{\alpha}f = Df$. *In particular, when* $\alpha = 0$, *then*

$$D_{a+}^{0}f = f.$$

Definition 2.2.4. [55, p. 70]: *Let* $\Omega = [a, b]$ $(-\infty < a < b < \infty)$ *be a finite interval on the real axis* $\mathbb{R}$. *The Riemann-Liouville right-sided fractional derivative* $D_{b-}^{\alpha}f$ *of order* α $(0 \leq \alpha < 1)$ *is defined by*

$$\left(D_{b-}^{\alpha}f\right)(x) = -\frac{d}{dx}\left(I_{b-}^{1-\alpha}f\right)(x)$$

that is

$$\left(D_{b-}^{\alpha}f\right) = -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_x^b \frac{f(t)}{(t-x)^{\alpha}}dt \ \left(a \leq x < b,\ 0 < \alpha < 1\right).$$

In particular, when $\alpha = 0$, *then*

$$D_{b-}^{0}f = f.$$

It can be directly verified that the Riemann-Liouville fractional integral and fractional derivative of the power function $(x-a)^{\beta-1}$ $(\beta > 0)$ yield the same power function with α added or subtracted from the power $\beta > 0$ with a certain coefficient in front of this power function.

Property 2.2.1. [55, p. 71]: *If* $\alpha \geq 0$ *and* $\beta > 0$, *then*

$$\left(I_{a+}^{\alpha}\left(t-a\right)^{\beta-1}\right)(x) = \frac{\Gamma\left(\beta\right)}{\Gamma\left(\beta+\alpha\right)}\left(x-a\right)^{\beta+\alpha-1}\ , \quad \alpha \geq 0$$

$$\left(D_{a+}^{\alpha}\left(t-a\right)^{\beta-1}\right)(x) = \frac{\Gamma\left(\beta\right)}{\Gamma\left(\beta-\alpha\right)}\left(x-a\right)^{\beta-\alpha-1}\ , \quad \alpha \geq 0.$$

In particular, if $\beta = 1$ *and* $\alpha \geq 0$, *then the Riemann-Liouville fractional derivative of a constant is not equal to zero:*

$$\left(D_{a+}^{\alpha}1\right)(x) = \frac{\left(x-a\right)^{-\alpha}}{\Gamma\left(1-\alpha\right)},\ \alpha \geq 0.$$

Property 2.2.2. [55, p. 72]: *If* $0 < \alpha \leq 1$, *then*

$$\left(D_{a+}^{\alpha}\left(t-a\right)^{\alpha-1}\right)(x) = 0.$$

Lemma 2.2.1. [60, p. 6]: *If* $0 \leq \gamma < 1$, *then the fractional integration operator* I_{a+}^{α} *of order* α $(\alpha > 0)$ *is bounded in* $C_{\gamma}\left[a,b\right]$ *:*

$$\left\|I_{a+}^{\alpha}g\right\|_{C_{\gamma}} \leq \frac{\Gamma\left(1-\gamma\right)}{\Gamma\left(1+\alpha-\gamma\right)}\left(b-a\right)^{\alpha}\left\|g\right\|_{C_{\gamma}},$$

here I_{a+}^{α} *is the Riemann-Liouville fractional integral operator and* $g \in C_{\gamma}\left[a,b\right]$.

Furthermore, we have the following important lemmas.

Lemma 2.2.2. [60, p. 6]: *The fractional integration operator* I_{a+}^{α} *of order* α $(\alpha > 0)$ *is a mapping from* $C[a,b]$ *to* $C[a,b]$, *and*

$$\left\| I_{a+}^{\alpha} g \right\|_{C} \leq \frac{(b-a)^{\alpha}}{\alpha \Gamma(\alpha)} \left\| g \right\|_{C},$$

where $g \in C[a,b]$.

Lemma 2.2.3. [60, p. 5]: *Let* $0 \leq \gamma < 1$, $a < c < b$, $g \in C_{\gamma}[a,c]$ *and* $g \in C[c,b]$. *Then* $g \in C_{\gamma}[a,b]$ *and*

$$\left\| g \right\|_{C_{\gamma}[a,b]} \leq \max \left[\left\| g \right\|_{C_{\gamma}[a,c]}, (b-a)^{\gamma} \left\| g \right\|_{C[c,b]} \right].$$

Now we consider some other properties of the Riemann-Liouville fractional integral (Definition 2.2.1) and the Riemann-Liouville fractional derivative (Definition 2.2.3) in the space $C_{\gamma}[a,b]$ and $C_{\gamma}^{n}[a,b]$ defined in Definition 2.1.4 and Definition 2.1.5, respectively. The existence of the fractional integral $I_{a+}^{\alpha} f$ in the space $C_{\gamma}[a,b]$ and the fractional derivative $D_{a+}^{\alpha} f$ in the space $C_{\gamma}^{n}[a,b]$ are given by the following lemmas.

Lemma 2.2.4. [55, p. 76]: *The following hold*

(a) Let $\alpha > 0$ *and* $0 \leq \gamma < 1$.

If $\gamma > \alpha$, *then the fractional integration operator* I_{a+}^{α} *is bounded from* $C_{\gamma}[a,b]$ *into* $C_{\gamma-\alpha}[a,b]$:

$$\left\| I_{a+}^{\alpha} f \right\|_{C_{\gamma-\alpha}} \leq k_1 \left\| f \right\|_{C_{\gamma}},$$

$$k_1 = \frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)}.$$

In particular I^{α}_{a+} *is bounded in* $C_{\gamma}[a,b]$.

(b) *If* $\gamma \le \alpha$, *then the fractional integration operator* I^{α}_{a+} *is bounded from* $C_{\gamma}[a,b]$ *into* $C[a,b]$:

$$\left\| I^{\alpha}_{a+} f \right\|_{C} \le k_2 \left\| f \right\|_{C_{\gamma}},$$

$$k_2 = (b-a)^{\alpha-\gamma} \frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)}.$$

In particular I^{α}_{a+} *is bounded in* $C_{\gamma}[a,b]$.

Lemma 2.2.5. [55, p. 77]: *Let* $0 < \alpha < 1$ *and* $0 \le \gamma < 1$. *If* $f \in C^{1}_{\gamma}$, *then the fractional derivatives* D^{α}_{a+} *and* D^{α}_{b-} *exist on* $(a,b]$ and $[a,b)$ respectively, *and can be represented in the forms*

$$\left(D^{\alpha}_{a^{+}} f\right)(x) = \frac{1}{\Gamma(1-\alpha)} \left[\frac{f(a)}{(x-a)^{\alpha}} + \int_{a}^{x} \frac{f'(t)\,dt}{(x-t)^{\alpha}} \right]$$

and

$$\left(D^{\alpha}_{b^{-}} f\right)(x) = \frac{1}{\Gamma(1-\alpha)} \left[\frac{f(b)}{(b-x)^{\alpha}} - \int_{x}^{b} \frac{f'(t)\,dt}{(t-x)^{\alpha}} \right],$$

Lemma 2.2.6. [55, p. 77]: (*The Semigroup Property of the fractional integration operator* I^{α}_{a+}) *Let* $\alpha > 0$, $\beta > 0$ *and* $0 \le \gamma < 1$. *If* $f \in C_{\gamma}[a,b]$ *then the equation*

$$I^{\alpha}_{a+} I^{\beta}_{a+} f = I^{\alpha+\beta}_{a+} f$$

holds at any point $x \in (a,b]$. *When* $f \in C[a,b]$ *this relation is valid at any point* $x \in [a,b]$.

The following assertion shows that the fractional differentiation is an inverse operation to the fractional integration from the left.

Lemma 2.2.7. [55, p. 77]: *Let* $\alpha > 0$ *and* $0 \leq \gamma < 1$. *If* $f \in C_\gamma[a,b]$, *then the relation*

$$D_{a+}^{\alpha} I_{a+}^{\alpha} f = f$$

holds at any point $x \in (a,b]$. *When* $f \in C[a,b]$ *this relation is valid at any point* $x \in [a,b]$.

Another composition property between the fractional differentiation operator (Definition 2.2.3) and the fractional integration (Definition 2.2.1) is given next.

Property 2.2.3. [55, p. 77]: *Let* $\alpha > \beta > 0$ *and* $0 \leq \gamma < 1$. *If* $f \in C_\gamma[a,b]$, *then the relation*

$$D_{a+}^{\beta} I_{a+}^{\alpha} f = I_{a+}^{\alpha-\beta} f$$

holds at any point $x \in (a,b]$. *When* $f \in C[a,b]$ *this relation is valid at any point* $x \in [a,b]$. *In particular, when* $\beta = k \in \mathbb{N}$ *and* $\alpha > k$, *then* $D_{a+}^{k} I_{a+}^{\alpha} f = I_{a+}^{\alpha-k} f$.

The following result provides another composition of the fractional integration operator I_{a+}^{α} with the fractional differentiation operator D_{a+}^{α}.

Lemma 2.2.8. [55, p. 77]: *Let* $0 < \alpha < 1$, $0 \leq \gamma < 1$. *Also let* $I_{a+}^{1-\alpha} f$ *be the fractional integral of the function* f *of order* $1-\alpha$.

If $f \in C_\gamma[a,b]$ *and* $I_{a+}^{1-\alpha} f \in C_\gamma^1[a,b]$, *then the equality*

$$\left(I_{a+}^{\alpha} D_{a+}^{\alpha} f\right)(x) = f(x) - \frac{\left(I_{a+}^{1-\alpha} f\right)(a)}{\Gamma(\alpha)} (x-a)^{\alpha-1},$$

holds at any point $x \in (a,b]$.

Lemma 2.2.9. [55, p. 76]: (Fractional Integration by Parts) : *Let* $\alpha > 0$, $p \geq 1$, $q \geq 1$ *and* $\frac{1}{p} + \frac{1}{q} \leq 1 + \alpha$ *(*$p \neq 1$ *and* $q \neq 1$ *in the case when* $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$*). If* $\varphi \in L_p(a,b)$ *and* $\psi \in L_q(a,b)$, *then*

$$\int_a^b \varphi(x) \left(I_{a+}^{\alpha}\psi\right)(x)\, dx = \int_a^b \psi(x) \left(I_{b-}^{\alpha}\varphi\right)(x)\, dx.$$

2.3 Hadamard Type Fractional Integrals and Fractional Derivatives

In this section we present the definitions and some properties of the Hadamard type fractional integrals and fractional derivatives which will be involved in the problems investigated later.

Definition 2.3.1. [55, p. 110]: *Let* (a,b) $(0 \leq a < b \leq \infty)$ *be a finite or infinite interval of the half–axis* $\mathbb{R}^+$ *and let* $\alpha > 0$*. The Hadamard left–sided fractional integral* $\mathcal{J}_{a^+}^{\alpha} f$ *of order* $\alpha > 0$ *is defined by*

$$\left(\mathcal{J}_{a^+}^{\alpha} f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \left(\log \frac{x}{t}\right)^{\alpha-1} \frac{f(t)\, dt}{t} \quad (a < x < b)$$

provided that the integral exists. When $\alpha = 0$, *we set*

$$\mathcal{J}_{a^+}^{0} f = f.$$

Definition 2.3.2. [55, p. 110]: *Let (a,b) $(0 \le a < b \le \infty)$ be a finite or infinite interval of the half–axis $\mathbb{R}^+$ and let $\alpha > 0$. The Hadamard right–sided fractional integral $\mathcal{J}_{b^-}^{\alpha} f$ of order $\alpha > 0$ is defined by*

$$\left(\mathcal{J}_{b^-}^{\alpha} f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \left(\log \frac{t}{x}\right)^{\alpha-1} \frac{f(t)\,dt}{t} \quad (a < x < b)$$

provided that the integral exists. When $\alpha = 0$, we set

$$\mathcal{J}_{b^-}^{0} f = f.$$

Definition 2.3.3. [55, p. 111]: *The left–sided Hadamard fractional derivative of order $0 \le \alpha < 1$ on (a,b) is defined by*

$$\left(\mathcal{D}_{a^+}^{\alpha} f\right)(x) := \delta\left(\mathcal{J}_{a^+}^{1-\alpha} f\right)(x),$$

that is

$$\left(\mathcal{D}_{a^+}^{\alpha} f\right)(x) = \left(x \frac{d}{dx}\right) \frac{1}{\Gamma(1-\alpha)} \int_a^x \left(\log \frac{x}{t}\right)^{-\alpha} \frac{f(t)\,dt}{t} \quad (a < x < b).$$

In particular, when $\alpha = 0$ we have

$$\mathcal{D}_{a^+}^{0} f = f.$$

Definition 2.3.4. [55, p. 111]: *The right–sided Hadamard fractional derivative of order α ($0 \leq \alpha < 1$) on (a, b) is defined by*

$$\left(\mathcal{D}_{b^-}^{\alpha} f\right)(x) := -\delta\left(\mathcal{J}_{b^-}^{1-\alpha} f\right)(x),$$

that is

$$\left(\mathcal{D}_{b^-}^{\alpha} f\right)(x) = -\left(x\frac{d}{dx}\right)\frac{1}{\Gamma(1-\alpha)}\int_x^b\left(\log\frac{t}{x}\right)^{-\alpha}\frac{f(t)\,dt}{t} \quad (a < x < b).$$

In particular, when $\alpha = 0$ we have

$$\mathcal{D}_{b^-}^{0} f = f.$$

It can be directly verified that the Hadamard fractional integral and fractional derivative of the logarithmic function $\left(\log\frac{x}{a}\right)^{\beta-1}$ yield the same logarithmic function with α added or subtracted from the power with a certain coefficient in front of the logarithmic function.

Property 2.3.1. [55, p. 112]: *If $\alpha > 0$, $\beta > 0$ and $0 < a < b < \infty$, then*

$$\left(\mathcal{J}_{a^+}^{\alpha}\left(\log\frac{t}{a}\right)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\left(\log\frac{x}{a}\right)^{\beta+\alpha-1},$$

and

$$\left(\mathcal{D}_{a^+}^{\alpha}\left(\log\frac{t}{a}\right)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left(\log\frac{x}{a}\right)^{\beta-\alpha-1}.$$

In particular, if $\beta = 1$ *and* $\alpha \geq 0$, *then the Hadamard fractional derivative of a constant is not equal to zero:*

$$\left(\mathcal{D}_{a+}^{\alpha} 1\right)(x) = \frac{1}{\Gamma(1-\alpha)} \left(\log \frac{x}{a}\right)^{-\alpha},$$

when $0 < \alpha < 1$.

Lemma 2.3.1. [55, p. 113]: *If* $0 < \alpha \leq 1$ *and* $0 < a < b < \infty$. *The equality* $\left(\mathcal{D}_{a+}^{\alpha} y\right)(x) = 0$ is valid

$$y(x) = c \left(\log \frac{x}{a}\right)^{\alpha-1}$$

for any $c \in \mathbb{R}$.

The Hadamard fractional integra (Definition 2.3.1) satisfies the following semigroup property.

Property 2.3.2. [55, p. 118]: *Let* $\alpha > 0$, $\beta > 0$ *and* $0 \leq \mu < 1$. *If* $0 < a < b < \infty$, *then, for* $f \in C_{\mu,\log}[a,b]$

$$\mathcal{J}_{a+}^{\alpha} \mathcal{J}_{a+}^{\beta} f = \mathcal{J}_{a+}^{\alpha+\beta} f$$

holds at any point $x \in (a,b]$. *When* $f \in C[a,b]$ *this relation is valid at any point* $x \in [a,b]$.

Compositions between the fractional differentiation and fractional integration operators are given in the following properties.

Property 2.3.3. [55, p. 118]: *Let* $\alpha > \beta > 0$. *If* $f \in C_{\mu,\log}[a,b]$ *and* $0 < a < b < \infty$ *then*

$$\mathcal{D}_{a+}^{\beta} \mathcal{J}_{a+}^{\alpha} f = \mathcal{J}_{a+}^{\alpha-\beta} f$$

holds at any point $x \in (a,b]$. *When* $f \in C[a,b]$ *this relation is valid at any point* $x \in [a,b]$.

In particular, when $\beta = k \in \mathbb{N}$ *and* $\alpha > k$, *then*

$$\mathcal{D}^{k}_{a^+}\mathcal{J}^{\alpha}_{a^+}f = \mathcal{J}^{\alpha-k}_{a^+}f.$$

The Hadamard fractional derivative is the left inverse operator to the corresponding fractional integral.

Property 2.3.4. [55, p. 116]: *Let* $\alpha > 0$. *If* $0 < a < b < \infty$ *and* $f \in C_{\mu,\log}[a,b]$ *then*

$$\mathcal{D}^{\alpha}_{a^+}\mathcal{J}^{\alpha}_{a^+}f = f, \quad x \in (a,b].$$

The following result provides a formula for the composition of the fractional differentiation operator $\mathcal{D}^{\alpha}_{a^+}$ with the fractional integration operator $\mathcal{J}^{\alpha}_{a^+}$. It shows that fractional differentiation is not the right inverse operator of the fractional integral in general.

Theorem 2.3.1. [55, p. 119]: *Let* $0 < \alpha < 1$ *and* $0 < a < b < \infty$. *Also let* $\mathcal{J}^{1-\alpha}_{a^+}f$ *be the Hadamard fractional integral of the order* $1-\alpha$ of the function f.

If $f \in C_{\mu,\log}[a,b]$ *and* $\mathcal{J}^{1-\alpha}_{a^+}f \in C^{1}_{\delta,\mu}[a,b]$, *then*

$$\left(\mathcal{J}^{\alpha}_{a^+}\mathcal{D}^{\alpha}_{a^+}f\right)(x) = f(x) - \frac{\left(\mathcal{J}^{1-\alpha}_{a^+}f\right)(a)}{\Gamma(\alpha)}\left(\log\frac{x}{a}\right)^{\alpha-1}.$$

holds at any point $x \in (a,b]$. *If* $f \in C[a,b]$ *and* $\mathcal{J}^{1-\alpha}_{a^+}f \in C^{1}_{\delta}[a,b]$, *then the relation holds at any point* $x \in [a,b]$.

Now we consider the properties of the Hadamard fractional integral and the Hadamard fractional derivative in the spaces $C_{\mu,\log}[a,b]$ and $C^1_{\delta,\mu}[a,b]$ (see Definitions 2.1.6 and 2.1.7). The existence of the fractional integral $\mathcal{J}^{\alpha}_{a+}f$ in the space $C_{\mu,\log}[a,b]$ and of the fractional derivative $\mathcal{D}^{\alpha}_{a+}y$ in the space $C^1_{\delta,\mu}[a,b]$ are given by the following assertions.

Lemma 2.3.2. [55, p. 118]: *Let $0 < a < b < \infty$, $\alpha > 0$ and $0 \le \mu < 1$.*

(a) If $\mu > \alpha > 0$, then the fractional integration operator $\mathcal{J}^{\alpha}_{a+}$ is bounded from $C_{\mu,\log}[a,b]$ into $C_{\mu-\alpha,\log}[a,b]$:

$$\|\mathcal{J}^{\alpha}_{a+}f\|_{C_{\mu-\alpha,\log}} \le k_1 \|f\|_{C_{\mu,\log}}$$

where

$$k_1 = \left(\log\frac{b}{a}\right)^{\alpha}\frac{\Gamma(1-\mu)}{\Gamma[1+\alpha-\mu]}.$$

In particular, $\mathcal{J}^{\alpha}_{a+}$ is bounded in $C_{\mu,\log}[a,b]$.

(b) If $\mu \le \alpha$, then the fractional integration operator $\mathcal{J}^{\alpha}_{a+}$ is bounded from $C_{\mu,\log}[a,b]$ into $C[a,b]$:

$$\|\mathcal{J}^{\alpha}_{a+}f\|_{C} \le k_2 \|f\|_{C_{\mu,\log}}$$

where

$$k_2 = \left(\log\frac{b}{a}\right)^{\alpha-\mu}\frac{\Gamma(1-\mu)}{\Gamma(1+\alpha-\mu)}.$$

In particular, $\mathcal{J}^{\alpha}_{a+}$ is bounded in $C_{\mu,\log}[a,b]$.

Lemma 2.3.3. [55, p. 118]: *Let $0 \le \alpha < 1$ and $0 \le \gamma < 1$. If $f \in C^1_{\gamma,\log}[a,b]$, then the fractional derivatives $\mathcal{D}^{\alpha}_{a+}$ and $\mathcal{D}^{\alpha}_{b-}$ exist on $(a,b]$ and $[a,b)$, respectively ($a > 0$)*

and can be represented in the forms

$$\left(\mathcal{D}_{a+}^{\alpha} f\right)(x) = \frac{f(a)}{\Gamma(1-\alpha)} \left(\log \frac{x}{a}\right)^{-\alpha} + \frac{1}{\Gamma(1-\alpha)} \int_{a}^{x} \left(\log \frac{x}{t}\right)^{-\alpha} f'(t)\, dt,$$

and

$$\left(\mathcal{D}_{b-}^{\alpha} f\right)(x) = \frac{f(b)}{\Gamma(1-\alpha)} \left(\log \frac{b}{x}\right)^{-\alpha} - \frac{1}{\Gamma(1-\alpha)} \int_{x}^{b} \left(\log \frac{t}{x}\right)^{-\alpha} f'(t)\, dt,$$

respectively.

Lemma 2.3.4. [55, p. 118]: *The Hadamard fractional integration operator $\mathcal{J}_{a+}^{\alpha}$ of order α ($\alpha > 0$) is a mapping from $C[a,b]$ to $C[a,b]$ and*

$$\left\| \mathcal{J}_{a+}^{\alpha} g \right\|_{C[a,b]} \leq \frac{\left(\log \frac{b}{a}\right)^{\alpha}}{\alpha \Gamma(\alpha)} \left\| g \right\|_{C[a,b]},$$

where $g \in C[a,b]$.

Lemma 2.3.5. [60, p. 7]: *Let $0 \leq \gamma < 1$, $0 < a < c < b < \infty$, $g \in C_{\gamma,\log}[a,c]$ and $g \in C[c,b]$. Then $g \in C_{\gamma,\log}[a,b]$ and*

$$\left\| g \right\|_{C_{\gamma,\log}[a,b]} \leq \max \left\{ \left\| g \right\|_{C_{\gamma,\log}[a,c]}, \left(\log \frac{b}{a}\right)^{\gamma} \left\| g \right\|_{C[c,b]} \right\}.$$

Lemma 2.3.6. [15, p. 13] (Fractional Integration by Parts): *Let $\alpha > 0$ and $1 \leq p \leq \infty$. If $\varphi \in L_p(\mathbb{R}^+)$ and $\psi \in X_{-1/p}^{q}$, then*

$$\int_{0}^{\infty} \varphi(x) \left(\mathcal{J}_{+}^{\alpha} \psi\right)(x) \frac{dx}{x} = \int_{0}^{\infty} \psi(x) \left(\mathcal{J}_{-}^{\alpha} \varphi\right)(x) \frac{dx}{x},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

2.4 Caputo Fractional Derivative

In this section we present the definitions of the Caputo fractional derivative ${}^cD^{\alpha}_{a+}f$ and Hadamard–Caputo fractional derivative ${}^c\mathcal{D}^{\alpha}_{a+}f$ of order α $(0 < \alpha < 1)$, respectively.

Definition 2.4.1: *Let* $[a, b]$ *be a finite interval of the real line* $\mathbb{R}$*. The fractional derivative* ${}^cD^{\alpha}_{a+}f$ *of order* α $(0 < \alpha < 1)$ *on* $[a, b]$ *defined by*

$$ {}^cD^{\alpha}_{a+}f = I^{1-\alpha}_{a+}Df $$

where $D = \frac{d}{dx}$*, is called the Caputo fractional derivative of* f *of order* α $(0 < \alpha < 1)$*.*

Definition 2.4.2: *Let* (a, b) *be a finite interval of the half–axis* $\mathbb{R}^+$*. The fractional derivative* ${}^c\mathcal{D}^{\alpha}_{a+}f$ *of order* α $(0 < \alpha < 1)$ *on* (a, b) *defined by*

$$ {}^c\mathcal{D}^{\alpha}_{a+}f = \mathcal{J}^{1-\alpha}_{a+}\delta f $$

where $\delta = x\frac{d}{dx}$*, is called the Hadamard–Caputo fractional derivative of order* α*.*

2.5 Some Important Results

In this section we present some other important definitions, lemmas, theorems and properties. These will determine the assumption, tools and the methods utilized in our results later.

Definition 2.5.1. [60, p. 3]: (*Lipschitz condition*) *Assume that* $f[., y(.)]$ *is defined on the set* $(a,b]\times\mathbf{G}$ $(\mathbf{G}\subset\mathbb{R})$. $f[., y(.)]$ *is said to satisfy the Lipschitz condition with respect to the second variable, if for all* $x\in(a,b]$ *and for any* $y_1, y_2\in\mathbf{G}\subset\mathbb{R}$ *one has*

$$\left|f[x,y_1]-f[x,y_2]\right|\le A\left|y_1-y_2\right|,$$

where $A>0$ *does not depend on* $x\in(a,b]$. *In this case we say that the function is Lipschitz continuous with respect to* y *with Lipschitz constant* A.

Lemma 2.5.1. [72, p. 23]: *If* λ, ν, $\omega>0$, *then for any* $t>0$, *we have*

$$\int_0^t (t-s)^{\nu-1}s^{\lambda-1}e^{-\omega s}ds\le Ct^{\nu-1},$$

where C *is a positive constant independent of* t. *In fact,*

$$C=\max\left\{1,2^{1-\nu}\right\}\Gamma(\lambda)\left(1+\lambda(\lambda+1)/\nu\right)\omega^{-\lambda}.$$

Lemma 2.5.2. [41, p. 581]: *We have, for positive* a, b *and* $\gamma>1$, *the inequality*

$$(a+b)^\gamma\le 2^{\gamma-1}\left(a^\gamma+b^\gamma\right).$$

Lemma 2.5.3. [81, p. 126]: *Let* a, b *be two continuous, positive functions defined on* $[t_0,\infty)$, $t_0\ge 0$, *and* $w:[0,\infty)\to[0,\infty)$ *be a continuous monotonic nondecreasing function such that* $w(0)=0$ *and* $w(x)>0$ *for* $x>0$. *If* u *is a positive differentiable* *function on* $[t_0, \infty)$ *that satisfies*

$$u'(t) \leq a(t)\, w(u(t)) + b(t), \quad t \in [t_0, \infty),$$

then we have

$$u(t) \leq G^{-1}\left[G\left(u(t_0) + \int_{t_0}^{t} b(s)\, ds\right) + \int_{t_0}^{t} a(s)\, ds\right],$$

for the values of t*, for which the right-hand side is well-defined, where*

$$G(r) = \int_{r_0}^{r} \frac{ds}{w(s)}, \quad r > r_0 > 0.$$

Theorem 2.5.1. [55, p. 68]: (*Banach Fixed Point Theorem) Let* (U, d) *be a nonempty complete metric space, let* $0 \leq w < 1$*, and let* $T : U \rightarrow U$ *be a map such that, for every* $u, v \in U$*, the relation*

$$d(Tu, Tv) \leq wd(u, v), \; 0 \leq w < 1$$

holds. Then the operator T *has a unique fixed point* $u^* \in U$*.*

Furthermore, if T^k $(k \in \mathbb{N})$ *is the sequence of operators defined by*

$$T^1 = T \text{ and } T^k = TT^{k-1} \,(k \in \mathbb{N} \backslash \{1\}),$$

then, for any $u_0 \in U$, *the sequence* $\left\{T^k u_0\right\}_{k=1}^{\infty}$ *converges to the above fixed point* u^*.

Theorem 2.5.2. [63, p. 1268] : (Young's inequality) *If* a *and* b *are nonnegative real numbers and* p *and* q *are positive real numbers such that* $1/p + 1/q = 1$ *then we have*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Equality holds if and only if $a^p = b^q$.

Chapter 3

Fractional Differential Problems with Hilfer-Riemann Fractional Derivative

This chapter is devoted to proving the existence and uniqueness of solutions to a Cauchy type problem of fractional order on a finite interval of the real axis in a weighted space of continuous functions.

Also a stability result and a non-existence result are stated and proved with Hilfer type fractional derivative.

We consider the Cauchy type problem

$$\begin{cases} \left(D_{a+}^{\alpha,\beta} y\right)(x) = f\left[x, y(x)\right], \ x > a, \quad 0 < \alpha < 1, \quad 0 \leq \beta \leq 1 \\ \left(I_{a+}^{(1-\beta)(1-\alpha)} y\right)(a) = c. \end{cases} \tag{3.1}$$

in the space $C_{1-\gamma}^{\alpha,\beta}[a,b]$ defined for $\gamma = \alpha + \beta - \alpha\beta$ $(0 < \alpha, \beta < 1)$ by

$$C_{1-\gamma}^{\alpha,\beta}[a,b] = \left\{ y \in C_{1-\gamma}[a,b], D_{a+}^{\alpha,\beta} y \in C_{1-\gamma}[a,b] \right\}, \tag{3.2}$$

and

$$C_{1-\gamma}^{\gamma}[a,b] = \left\{ y \in C_{1-\gamma}[a,b], D_{a+}^{\gamma} y \in C_{1-\gamma}[a,b] \right\}.$$

We recall that $C_{1-\gamma}[a,b]$ is the weighted space of continuous functions on $(a,b]$

$$C_{1-\gamma}[a,b] = \left\{ g : (a,b] \to \mathbb{R}\colon (x-a)^{1-\gamma} g(x) \in C[a,b] \right\}, \tag{3.3}$$

and

$$\left(D_{a+}^{\alpha,\beta} y\right)(x) = \left(I_{a+}^{\beta(1-\alpha)} \frac{d}{dx} I_{a+}^{(1-\beta)(1-\alpha)} y\right)(x)$$

is the Hilfer fractional derivative of order $0 < \alpha < 1$ and type $0 \leq \beta \leq 1$.

Our investigations are based on reducing the considered problem to a Volterra integral equation of the second kind

$$y(x) = \frac{c(x-a)^{(\alpha-1)(1-\beta)}}{\Gamma(\alpha+\beta-\alpha\beta)} + \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f[t, y(t)]\, dt, \; x > a, \tag{3.4}$$

and on using the Banach fixed point theorem.

3.1 Equivalence of the Cauchy Type Problem and the Volterra Integral Equation

In this section we prove the equivalence between the Cauchy type problem (3.1) and the nonlinear Volterra integral equation (3.4) in the sense that, if $y \in C^{\gamma}_{1-\gamma}[a,b]$ satisfies one of these relations, then it also satisfies the other one. To establish such a result, we assume that a function $f[., y(.)]$ belongs to $C_{1-\gamma}[a,b]$ for any $y \in C_{1-\gamma}[a,b]$. For this we need the auxiliary assertion.

Lemma 3.1.1: *Let* $\alpha > 0$, $0 \leq \gamma < 1$ *and* $f \in C_{\gamma}[a,b]$

If $\gamma < \alpha$ *then*

$$\left(I^{\alpha}_{a+} f\right)(a) = \lim_{x \to a} \left(I^{\alpha}_{a+} f\right)(x) = 0, \quad 0 \leq \gamma < \alpha.$$

Proof : Since $f \in C_{\gamma}[a,b]$ then $(x-a)^{\gamma} f(x)$ is continuous on $[a,b]$ and on $[a,b]$ we have

$$\left|(x-a)^{\gamma} f(x)\right| < M,$$

for some positive constant M.

Therefore

$$-M\left(I_{a+}^{\alpha}(t-a)^{-\gamma}\right)(x) < \left(I_{a+}^{\alpha} f\right)(x) < M\left(I_{a+}^{\alpha}(t-a)^{-\gamma}\right)(x)$$

and by using Property 2.2.1 (with $\beta = 1-\gamma > 0$) we have

$$-M\tfrac{\Gamma(1-\gamma)}{\Gamma(\alpha+1-\gamma)}(x-a)^{\alpha-\gamma} \quad < \quad \left(I_{a+}^{\alpha} f\right)(x) \quad < \quad M\tfrac{\Gamma(1-\gamma)}{\Gamma(\alpha+1-\gamma)}(x-a)^{\alpha-\gamma} \ .$$

As $\alpha > \gamma$ we see that

$$\left(I_{a+}^{\alpha} f\right)(a) = \lim_{x \to a}\left(I_{a+}^{\alpha} f\right)(x) = 0, \quad 0 \leq \gamma < \alpha$$

which completes the proof of Lemma 3.1.1.

Theorem 3.1.1: *Let* $\gamma = \alpha + \beta - \alpha\beta$ *where* $0 < \alpha < 1$ *and* $0 \leq \beta \leq 1$. *Let* $f : (a,b] \times \mathbb{R} \to \mathbb{R}$ *be a function such that* $f[., y(.)] \in C_{1-\gamma}[a,b]$ *for any* $y \in C_{1-\gamma}[a,b]$.

If $y \in C_{1-\gamma}^{\gamma}[a,b]$, *then* y *satisfies the (CFDP) (3.1) if and only if* y *satisfies the (IE) (3.4).*

Proof : First we prove the **necessity.** Let $y \in C_{1-\gamma}^{\gamma}[a,b]$ be a solution of problem (3.1). We want to prove that y is also a solution of the integral equation (3.4). It is clear, by the definition $C_{1-\gamma}^{\gamma}[a,b]$ above and Definition 2.2.3, that

$$D_{a+}^{\gamma}y = D\left(I_{a+}^{1-\gamma}y\right) \in C_{1-\gamma}[a,b], \qquad I_{a+}^{0}y = y,$$

and by Lemma 2.2.4(b) we have $I_{a+}^{1-\gamma}y \in C[a,b]$ since $y \in C_{1-\gamma}[a,b]$. Then by Definition 2.1.5 we have

$$I_{a+}^{1-\gamma}y \in C_{1-\gamma}^{1}[a,b].$$

Thus we can apply Lemma 2.2.8 to get

$$\left(I_{a+}^{\gamma}D_{a+}^{\gamma}y\right)(x) = y(x) - \frac{\left(I_{a+}^{1-\gamma}y\right)(a)}{\Gamma(\gamma)}(x-a)^{\gamma-1} \tag{3.5}$$

or

$$\left(I_{a+}^{\gamma}D_{a+}^{\gamma}y\right)(x) = y(x) - \frac{c}{\Gamma(\gamma)}(x-a)^{\gamma-1}, \tag{3.6}$$

where c comes from the initial condition in (3.1). By our hypothesis $f[.,y(.)] \in C_{1-\gamma}[a,b]$, and Lemma 2.2.1, we see that $I_{a+}^{\alpha}f \in C_{1-\gamma}[a,b]$. Applying the operator I_{a+}^{α} to both sides of (3.1) we get

$$I_{a+}^{\alpha}I_{a+}^{\beta(1-\alpha)}\left(D_{a+}^{\gamma}y\right) = I_{a+}^{\alpha}f[x,y(x)].$$

As we can apply lemma 2.2.6 to get

$$\left(I_{a+}^{\alpha+\beta(1-\alpha)}D_{a+}^{\gamma}y\right)(x) = I_{a+}^{\alpha}f[x,y(x)]$$

or

$$\left(I_{a+}^{\gamma}D_{a+}^{\gamma}y\right)(x) = \left(I_{a+}^{\alpha}f\left[t, y\left(t\right)\right]\right)(x),\ x \in (a, b]. \tag{3.7}$$

From (3.6) and (3.7) we obtain

$$y(x) = \frac{c}{\Gamma(\gamma)}(x-a)^{\gamma-1} + \left(I_{a+}^{\alpha}f\left[t, y\left(t\right)\right]\right)(x)$$

which is the equation (3.4), where $\gamma = \alpha + \beta - \alpha\beta$, and hence the necessity is proved. Now we prove the **sufficiency**. Let $y \in C_{1-\gamma}^{\gamma}[a, b]$ satisfy the equation (3.4), then $D_{a+}^{\gamma}y$ exists and $D_{a+}^{\gamma}y \in C_{1-\gamma}[a, b]$. Applying the operator D_{a+}^{γ} to both sides of (3.4) we find

$$\left(D_{a+}^{\gamma}y\right)(x) = \frac{c}{\Gamma(\gamma)}\left(D_{a+}^{\gamma}(t-a)^{\gamma-1}\right)(x) + \left(D_{a+}^{\gamma}I_{a+}^{\alpha}f\left[t, y\left(t\right)\right]\right)(x).$$

By using Property 2.2.2 and Definition 2.2.3 when $0 < \gamma < 1$, we have

$$\begin{aligned}\left(D_{a+}^{\gamma}y\right)(x) &= \tfrac{d}{dx}\left(I_{a+}^{1-\gamma}I_{a+}^{\alpha}f\left[t, y\left(t\right)\right]\right)(x)\\ &= \tfrac{d}{dx}\left(I_{a+}^{1-\beta(1-\alpha)}f\left[t, y\left(t\right)\right]\right)(x)\\ &= \left(D_{a+}^{\beta(1-\alpha)}f\left[t, y\left(t\right)\right]\right)(x).\end{aligned} \tag{3.8}$$

From (3.8) and $D_{a+}^{\gamma}y \in C_{1-\gamma}[a, b]$, we obtain that

$$\left(D_{a+}^{\beta(1-\alpha)}f\left[t, y\left(t\right)\right]\right) \in C_{1-\gamma}[a, b].$$

Applying the operator $I_{a+}^{\beta(1-\alpha)}$ to both sides of (3.8) we get

$$\left(I_{a+}^{\beta(1-\alpha)} D_{a+}^{\gamma} y\right)(x) = \left(I_{a+}^{\beta(1-\alpha)} D_{a+}^{\beta(1-\alpha)} f\left[t, y\left(t\right)\right]\right)(x).$$

That is

$$I_{a+}^{\beta(1-\alpha)} \frac{d}{dx}\left(I_{a+}^{1-\gamma} y\right)(x) = \left(I_{a+}^{\beta(1-\alpha)} D_{a+}^{\beta(1-\alpha)} f\left[t, y\left(t\right)\right]\right)(x).$$

By virtue of $\frac{d}{dx}\left(I_{a+}^{1-\beta(1-\alpha)} f\left[t, y\left(t\right)\right]\right) = \left(D_{a+}^{\beta(1-\alpha)} f\left[t, y\left(t\right)\right]\right) \in C_{1-\gamma}\left[a, b\right]$ and Lemma 2.2.4(b) we have $\left(I_{a+}^{1-\beta(1-\alpha)} f\left[t, y\left(t\right)\right]\right) \in C\left[a, b\right]$ since $f\left[., y\left(.\right)\right] \in C_{1-\gamma}\left[a, b\right]$ with $1-\gamma \leq 1-\beta\left(1-\alpha\right)$. Hence by Definition 2.1.5 we have

$$\left(I_{a+}^{1-\beta(1-\alpha)} f\left[t, y\left(t\right)\right]\right) \in C_{1-\gamma}^{1}\left[a, b\right].$$

Then Lemma 2.2.8 allows us to write

$$\left(D_{a+}^{\alpha,\beta} y\right)(x) = f\left[x, y\left(x\right)\right] - \frac{\left(I_{a+}^{1-\beta(1-\alpha)} f\left[t, y\left(t\right)\right]\right)(a)}{\Gamma\left[\beta\left(1-\alpha\right)\right]}\left(x-a\right)^{\beta(1-\alpha)-1}. \tag{3.9}$$

Lemma 3.1.1 (with γ replaced by $1-\gamma$, α by $1-\beta\left(1-\alpha\right)$ and $\left(1-\gamma\right) < 1-\beta\left(1-\alpha\right)$) implies that

$$\left(I_{a+}^{1-\beta(1-\alpha)} f\left[t, y\left(t\right)\right]\right)(a) = 0.$$

Hence the relation (3.9) reduces to

$$\left(D_{a+}^{\alpha,\beta} y\right) = f\left[x, y(x)\right].$$

Now we show that the initial condition in (3.1) also holds. To this end we apply the operator $I_{a+}^{1-\gamma}$ to both sides of (3.4)

$$\left(I_{a+}^{1-\gamma}y\right)(x)=\frac{c}{\Gamma(\gamma)}\left(I_{a+}^{1-\gamma}(t-a)^{\gamma-1}\right)(x)+\left(I_{a+}^{1-\gamma}I_{a+}^{\alpha}f[t,y(t)]\right)(x)$$

and use the Property 2.2.1 (with α replaced by $1-\gamma$ and β by γ) and Lemma 2.2.6 to obtain

$$\left(I_{a+}^{1-\gamma}y\right)(x)=c+\left(I_{a+}^{1-\beta(1-\alpha)}f[t,y(t)]\right)(x). \tag{3.10}$$

In (3.10), taking the limit as $x\to a$, we obtain

$$\left(I_{a+}^{1-\gamma}y\right)(a)=c+\left(I_{a+}^{1-\beta(1-\alpha)}f[t,y(t)]\right)(a).$$

But $\left(I_{a+}^{1-\beta(1-\alpha)}f[t,y(t)]\right)(a)=0$ since $f[.,y(.)]\in C_{1-\gamma}[a,b]$ (See Lemma 3.1.1) with γ replaced by $1-\gamma$ and α by $1-\beta(1-\alpha)$ where $1-\gamma<1-\beta(1-\alpha)$). Therefore

$$\left(I_{a+}^{1-\gamma}y\right)(a)=c$$

and sufficiency is proved.

3.2 Existence and Uniqueness of the Global Solution to the Cauchy Type Problem

In this section we establish the existence of a unique solution to the Cauchy type problem (3.1) in the space $C_{1-\gamma}^{\alpha,\beta}[a,b]$ defined in (3.2) under the conditions of Theorem 3.1.1 and an additional Lipschitz condition on $f[.,y(.)]$ with respect to the second variable: for all $x \in (a,b]$ and for all $y_1,\ y_2 \in \mathbf{G} \subset \mathbb{R}$,

$$|f[x,y_1] - f[x,y_2]| \leq A\,|y_1 - y_2| \qquad (A > 0)\,, \tag{3.11}$$

where $A > 0$ does not depend on $x \in (a,b]$.

We use the Banach fixed point theorem (see Theotem 2.5.1) to prove the existence and uniqueness in the appropriate space $C_{1-\gamma}^{\alpha,\beta}[a,b]$ where $\gamma = \alpha + \beta - \alpha\beta$.

Theorem 3.2.1: *Let $\gamma = \alpha + \beta - \alpha\beta$ where $(0 < \alpha < 1,\ 0 \leq \beta \leq 1)$. Let $f : (a,b] \times \mathbb{R} \rightarrow \mathbb{R}$ be a function such that $f[.,y(.)] \in C_{1-\gamma}[a,b]$ for any $y \in C_{1-\gamma}[a,b]$ and the Lipschitzian condition (3.11) holds with respect to the second variable.*

Then there exists a unique solution y for the Cauchy type problem (3.1) in the space $C_{1-\gamma}^{\alpha,\beta}[a,b]$.

Proof : First we prove the existence of a unique solution y in the space $C_{1-\gamma}[a,b]$. According to Theorem 3.1.1**,** it suffices to prove the existence of a unique solution $y \in C_{1-\gamma}[a,b]$ to the nonlinear Volterra integral equation (3.4). Let us select x_1 such that the inequality

$$w_1 = \frac{A\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}(x_1 - a)^{\alpha} < 1, \tag{3.12}$$

where $A > 0$ is the Lipschitz constant in (3.11), is satisfied. We start by proving that a unique solution $y \in C_{1-\gamma}[a, x_1]$ to equation (3.4) exists on the interval $(a, x_1]$. For this, we use the Banach fixed point theorem for the space$C_{1-\gamma}[a, x_1]$, which is a complete metric space with the distance given by

$$d(y_1, y_2) = \|y_1 - y_2\|_{C_{1-\gamma}[a,x_1]} := \max_{x\in[a,x_1]} \left|(x-a)^{1-\gamma}[y_1(x) - y_2(x)]\right|.$$

The integral equation (3.4) takes the form

$$y(x) = (Ty)(x) \tag{3.13}$$

where

$$(Ty)(x) = y_0(x) + \left(I^{\alpha}_{a+} f[t, y(t)]\right)(x) \tag{3.14}$$

with

$$y_0(x) = \frac{c}{\Gamma(\gamma)}(x-a)^{\gamma-1}. \tag{3.15}$$

Notice that T maps $C_{1-\gamma}[a, x_1]$ to itself. Since y_0 is clearly in $C_{1-\gamma}[a, x_1]$ and $\left(I^{\alpha}_{a+} f[t, y(t)]\right)$ also is in $C_{1-\gamma}[a, x_1]$ by Lemma 2.2.1 when $y \in C_{1-\gamma}[a, x_1]$. Therefore $Ty \in C_{1-\gamma}[a, x_1]$.

Moreover T is a contraction, that is

$$\|Ty_1 - Ty_2\|_{C_{1-\gamma}[a,x_1]} \leq w_1 \|y_1 - y_2\|_{C_{1-\gamma}[a,x_1]}\,,\ 0 < w_1 < 1. \tag{3.16}$$

This follows from (3.14), (3.11) and Lemma 2.2.1 as follows

$$\begin{aligned}
\|Ty_1 - Ty_2\|_{C_{1-\gamma}[a,x_1]} &= \left\| I_{a+}^{\alpha} f\left[t, y_1(t)\right] - I_{a+}^{\alpha} f\left[t, y_2(t)\right] \right\|_{C_{1-\gamma}[a,x_1]} \\
&\leq \left\| I_{a+}^{\alpha} \left[\left| f[t, y_1(t)] - f[t, y_2(t)] \right| \right] \right\|_{C_{1-\gamma}[a,x_1]} \\
&\leq (x_1 - a)^{\alpha} \tfrac{\Gamma(\gamma)}{\Gamma(\alpha+\gamma)} \|f[t, y_1(t)] - f[t, y_2(t)]\|_{C_{1-\gamma}[a,x_1]} \\
&\leq A\,(x_1 - a)^{\alpha} \tfrac{\Gamma(\gamma)}{\Gamma(\alpha+\gamma)} \|y_1(t) - y_2(t)\|_{C_{1-\gamma}[a,x_1]} \\
&\leq w_1 \|y_1(t) - y_2(t)\|_{C_{1-\gamma}[a,x_1]}\,.
\end{aligned}$$

Our assumption (3.12) allows us to apply the Banach fixed point Theorem (Theorem 2.5.1) to obtain a unique solution $y^* \in C_{1-\gamma}[a, x_1]$ to the equation (3.4) on the interval $[a, x_1]$.

Theorem 2.5.1 tells us that this solution y^* is a limit of a convergent sequence $T^m y_0^*$:

$$\lim_{m\to\infty} \|T^m y_0^* - y^*\|_{C_{1-\gamma}[a,x_1]} = 0,$$

where y_0^* is any function in $C_{1-\gamma}[a, x_1]$ and

$$\left(T^m y_0^*\right)(x) = \left(T T^{m-1} y_0^*\right)(x) = y_0(x) + \left(I_{a+}^{\alpha} f\left[t, \left(T^{m-1} y_0^*\right)(t)\right]\right)(x)\,,\ m \in \mathbb{N}$$

It is convenient to take $y_0^*(x) = y_0(x)$ with $y_0(x)$ defined by (3.15). If we denote by

$$y_m(x) := (T^m y_0^*)(x), \; m \in \mathbb{N}$$

then clearly

$$\lim_{m\to\infty} \|y_m - y^*\|_{C_{1-\gamma}[a,x_1]} = 0.$$

Next, we consider the interval $[x_1, b]$. From the equation (3.4) we have

$$\begin{aligned} y(x) &= \tfrac{c}{\Gamma(\gamma)}(x-a)^{\gamma-1} + \left(I_{a+}^{\alpha} f[t, y(t)]\right)(x) \\ &= \tfrac{c}{\Gamma(\gamma)}(x-a)^{\gamma-1} + \tfrac{1}{\Gamma(\alpha)}\textstyle\int_a^x (x-t)^{\alpha-1} f[t, y(t)]dt \\ &= \tfrac{c}{\Gamma(\gamma)}(x-a)^{\gamma-1} + \tfrac{1}{\Gamma(\alpha)}\textstyle\int_a^{x_1} (x-t)^{\alpha-1} f[t, y(t)]dt \\ &\quad + \tfrac{1}{\Gamma(\alpha)}\textstyle\int_{x_1}^x (x-t)^{\alpha-1} f[t, y(t)]dt. \end{aligned}$$

Since the function $y(t)$ is uniquely defined on the interval $(a, x_1]$, the last integral can be considered as the unknown function, and we rewrite the last equation as

$$y(x) = y_{01}(x) + \frac{1}{\Gamma(\alpha)} \int_{x_1}^x (x-t)^{\alpha-1} f[t, y(t)]dt, \tag{3.17}$$

where $y_{01}(x)$ is defined by

$$y_{01}(x) = \frac{c}{\Gamma(\gamma)}(x-a)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_a^{x_1} (x-t)^{\alpha-1} f[t, y^*(t)]dt, \tag{3.18}$$

and is a known function. We note that $y_{01}(x) \in C[x_1, b]$.

Next we prove the existence of a unique solution $y \in C[x_1, b]$ to equation (3.4) on the interval $[x_1, b]$. For this, we also use Banach fixed point theorem for the space $C[x_1, x_2]$, where $x_2 \in (x_1, b]$ satisfies

$$w_2 = \frac{A}{\alpha\Gamma(\alpha)} (x_2 - x_1)^{\alpha} < 1. \tag{3.19}$$

$C[x_1, x_2]$ is a complete metric space with the distance given by

$$d(y_1, y_2) = \|y_1 - y_2\|_{C[x_1,x_2]} := \max_{x\in[x_1,x_2]} |y_1(x) - y_2(x)|.$$

The integral equation (3.17) may be written shortly as

$$y(x) = (Ty)(x), \tag{3.20}$$

where the operator (again denoted by T) is given by

$$(Ty)(x) = y_{01}(x) + \frac{1}{\Gamma(\alpha)} \int_{x_1}^{x} (x-t)^{\alpha-1} f[t, y(t)]dt. \tag{3.21}$$

As in the first part of this proof, since $y_{01}(x) \in C[x_1, x_2]$ and $f[x, y(x)] \in C[x_1, x_2]$ for any $y \in C[x_1, x_2]$, then, by Lemma 2.2.2, We deduce that the integral in the right–hand side of (3.21) also belongs to $C[x_1, x_2]$, and hence $Ty \in C[x_1, x_2]$.

Moreover, using the Lipschitz condition (3.11) and applying Lemma 2.2.2, we find

$$\begin{aligned}
\|Ty_1 - Ty_2\|_{C[x_1,x_2]} &= \left\| I^{\alpha}_{x_1^+} f\left[t, y_1(t)\right] - I^{\alpha}_{x_1^+} f\left[t, y_2(t)\right] \right\|_{C[x_1,x_2]} \\
&\leq \left\| I^{\alpha}_{x_1^+} \left[\left| f[t, y_1(t)] - f[t, y_2(t)] \right| \right] \right\|_{C[x_1,x_2]} \\
&\leq \tfrac{1}{\alpha\Gamma(\alpha)} (x_2 - x_1)^{\alpha} \left\| f[t, y_1(t)] - f[t, y_2(t)] \right\|_{C[x_1,x_2]} \\
&\leq \tfrac{A}{\alpha\Gamma(\alpha)} (x_2 - x_1)^{\alpha} \left\| y_1(t) - y_2(t) \right\|_{C[x_1,x_2]} \\
&\leq w_2 \left\| y_1(t) - y_2(t) \right\|_{C[x_1,x_2]} .
\end{aligned}$$

This, together with our assumption $0 < w_2 < 1$, shows that T is a contraction and therefore from Theorem 2.5.1, there exists a unique solution $y_1^*(x) \in C[x_1, x_2]$ to equation (3.20) and hence to (3.4) on the interval $[x_1, x_2]$. Notice that $y_1^*(x_1) = y^*(x_1) = y_{01}(x_1)$. Furthermore Theorem 2.5.1, guarantees that this solution is the limit of a convergent sequence $T^m y_{01}^*$:

$$\lim_{m\to\infty} \|T^m y_{01}^* - y_1^*\|_{C[x_1,x_2]} = 0$$

where y_{01}^* is any function in $C[x_1, x_2]$, which we can pick $y_{01}^*(x) = y_{01}(x)$ defined by (3.18), if $y_0(x) \neq 0$ on $[x_1, x_2]$ we can take $y_{01}^*(x) = y_0(x)$ defined by (3.15). Therefore

$$\lim_{m\to\infty} \|y_m - y_1^*\|_{C[x_1,x_2]} = 0,$$

where

$$y_m(x) = (T^m y_{01}^*)(x) = y_{01}(x) + \frac{1}{\Gamma(\alpha)} \int_{x_1}^{x} (x-t)^{\alpha-1} f[t, \left(T^{m-1} y_{01}^*\right)(t)] dt.$$

If $x_2 \neq b$, we consider the interval $[x_2, x_3]$, such that $x_3 \leq b$ and

$$w_3 = \frac{A}{\alpha\Gamma(\alpha)}(x_3 - x_2)^\alpha < 1.$$

Using the same arguments as above, we derive that there exists a unique solution $y_2^*(x) \in C[x_2, x_3]$ to equation (3.4) on the interval $[x_2, x_3]$. If $x_3 \neq b$, then we continue the process until we reach a solution $y(x)$ to equation(3.4), $y(x) = y_k^*(x)$, and $y_k^*(x) \in C[x_k, x_{k+1}]$ $(k = 1, ..., L)$, where $a = x_0 < x_1 < ... < x_{L+1}$ and

$$w_{k+1} = \frac{A}{\alpha\Gamma(\alpha)}(x_{k+1} - x_k)^\alpha < 1,$$

and we take $y_0(x) = y_{0k}(x)$, and $y_0^*(x) = y_{0k}^*(x)$ $(k = 1, ..., L)$ on each interval $[x_k, x_{k+1}]$. Assume that $b - a > \frac{\Gamma(\alpha+\gamma)}{A\Gamma(\gamma)}$ (for otherwise take $x_1 = b$). Then, divide the length of the interval $\left[a + \frac{\Gamma(\alpha+\gamma)}{A\Gamma(\gamma)}, b\right]$ by $\left(\frac{\alpha\Gamma(\alpha)}{A}\right)^{1/\alpha}$. Let M be that quotient. It is clear that $L = [M] + 1$ and b is reached after a finite number of steps, $x_{L+1} = b$. Then, there exists a unique solution $y(x) \in C[x_1, b]$ to equation (3.4) on the interval $[x_1, b]$.

Considering both parts of the solution in $[a, x_1]$ and $[x_1, b]$ as one single part on $[a, b]$ and taking into account Lemma 2.2.3, we obtain that there exists a unique solution $y(x) \in C_{1-\gamma}[a, b]$ to the Volterra integral equation (3.4) on the whole interval $[a, b]$, and hence $y \in C_{1-\gamma}[a, b]$ is the unique solution to the Cauchy-type problem (3.1).

It remains to show that such a unique solution $y \in C_{1-\gamma}[a,b]$ is actually in $C_{1-\gamma}^{\alpha,\beta}[a, b]$. To this end we need to prove that $D_{a+}^{\alpha,\beta} y \in C_{1-\gamma}[a,b]$. Let us recall that our y is a limit of the sequence $y_m(x)$, where $y_m(x) = (T^m y_0^*) \in C_{1-\gamma}[a,b]$ (where T is defined by (3.14) for $[a, x_1]$, by (3.21) for $[x_1, x_2]$, etc...), that is

$$\lim_{m\to\infty} \|y_m - y\|_{C_{1-\gamma}[a,b]} = 0,$$

with the choice of certain y_0^* on each $[a, x_1]$,...,$[x_L, b]$. If $y_0 \neq 0$, then we can take $y_0^* = y_0$.

The equation in (3.1) and the Lipschitz condition (3.11), imply that

$$\begin{aligned} \left\| D_{a+}^{\alpha,\beta} y_m - D_{a+}^{\alpha,\beta} y \right\|_{C_{1-\gamma}[a,b]} &= \|f[x, y_m] - f[x, y]\|_{C_{1-\gamma}[a,b]} \\ &\leq A \|y_m - y\|_{C_{1-\gamma}[a,b]} \end{aligned}$$

and therefore

$$\lim_{m\to\infty} \left\| D_{a+}^{\alpha,\beta} y_m - D_{a+}^{\alpha,\beta} y \right\|_{C_{1-\gamma}[a,b]} = 0.$$

We entail from this relation that $D_{a+}^{\alpha,\beta} y \in C_{1-\gamma}[a,b]$ if $D_{a+}^{\alpha,\beta} y_m \in C_{1-\gamma}[a,b]$, $m = 1, 2, ...$

This latter property holds from the relation

$$(T^m y_0^*)(x) = y_0(x) + \left(I_{a+}^{\alpha} f\left[t, \left(T^{m-1} y_0^*\right)(t)\right]\right)(x), \ m \in \mathbb{N}$$

and the fact that $f[x, y(x)] \in C_{1-\gamma}[a,b]$ for any $y \in C_{1-\gamma}[a,b]$. This completes the proof of Theorem 3.2.1.

3.3 Stability of Hilfer Fractional Derivative

In this section we are consider with the following fractional differential problem with weighted initial data

$$\begin{cases} D_{0^+}^{\alpha,\beta} u(t) = f[t, u(t)], \quad t > 0, \\ t^{(1-\beta)(1-\alpha)} u(t) \,|_{t=0} = b, \end{cases} \tag{3.22}$$

where $D_{0^+}^{\alpha,\beta}$ is the Hilfer fractional derivative (**HFD**) of order $0 < \alpha < 1$ and type $0 \leq \beta \leq 1$, $b \in \mathbb{R}^*$ (the set of all real numbers except 0) and f is a continuous nonlinear function with respect to both of its arguments. We exploit the "initial decay" that is the behavior of solutions nearby the initial point and determine sufficient conditions on the nonlinearity which allow us to push and keep this behavior for all time in case of global existence.

We will assume the following hypotheses on the function $f(t, u)$:

(F) $f(t, u)$ is a continuous nonlinear function on $\mathbb{R}^+ \times \mathbb{R}$ and is such that

$$|f[t, u(t)]| \leq t^{\mu} e^{-\sigma t} \varphi(t) |u|^m, \quad \mu \geq 0, \quad m > 1, \quad \sigma > 0,$$

where φ is a continuous function on $\mathbb{R}^+$. As $f[., y(.)]$ is continuous, under the assumption of Theorem 3.2.1 we have existence and uniqueness in the space $C_{1-\gamma}^{\alpha,\beta}[0, T]$ for any $T > 0$.

Let p and q be conjugate exponents, i.e. $pq = p + q$ and $\lambda_1 := 1 + p[\mu - (1-\gamma)m]$,

$\lambda_2 := 1 + p(\alpha - 1)$ where $\gamma := \alpha + \beta - \alpha\beta$. If $\mu - (m-1)(1-\gamma) > 0$ and $q > 1/\alpha$, then $\lambda_1 > 0$ and $\lambda_2 > 0$. We denote by $\mathbf{L}$ the positive real number

$$\mathbf{L} := \frac{1}{(m-1)\, 2^{m(mq-1)} \left|b\right|^{mq(m-1)}} \left[\frac{\Gamma^p(\alpha)\,(\sigma p)^{\lambda_1}}{2^{p(1-\alpha)}\Gamma(\lambda_1)\,(1+\lambda_1/\lambda_2)} \right]^{mq/p}.$$

Theorem 3.3.1: *Suppose that the hypotheses of Theorem 3.2.1 holds,* $f[t, u(t)]$ *satisfies* ***(F)*** *and* $\mu - (m-1)(1-\gamma) > 0$. *If* $\varphi(t)$, $\varphi(t)\, t^{-m\beta(1-\alpha)} \in L^q(0,\infty)$ *for some* $q > 1/\alpha$, *then there exists a positive constant* C *such that the solution of (3.22) satisfies* $|u(t)| \le Ct^{\gamma-1}$, $t > 0$ *where* $\gamma = \alpha + \beta - \alpha\beta$, *provided that*

$$\|\varphi\|_q^{(m-1)q} \left(\int_0^\infty s^{-qm\beta(1-\alpha)} \varphi^q(s)\, ds \right) < \mathbf{L}.$$

Proof : Let us consider the Volterra integral equation

$$u(t) = bt^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f[s, u(s)]\, ds, \qquad t > 0, \tag{3.23}$$

associated to problem (3.22). Multiplying both sides of (3.23) by $t^{1-\gamma}$ and using the assumption ***(F)*** on $f([t, u(t)])$, we get

$$t^{1-\gamma}|u(t)| \le |b| + \frac{t^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{\mu} e^{-\sigma s} \varphi(s) |u(s)|^m\, ds, \qquad t > 0. \tag{3.24}$$

Let v denote the left-hand side of (3.24). Inserting the term $s^{(1-\gamma)m} s^{-(1-\gamma)m}$ inside the integral gives

$$v(t) \leq |b| + \frac{t^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{\mu-(1-\gamma)m} e^{-\sigma s} \varphi(s) v^m(s)\, ds, \qquad t > 0. \qquad (3.25)$$

Now the Hölder inequality with exponents p and q yields

$$\int_0^t (t-s)^{\alpha-1} s^{\mu-(1-\gamma)m} e^{-\sigma s} \varphi(s) v^m(s)\, ds$$

$$\leq \left(\int_0^t (t-s)^{p(\alpha-1)} s^{p[\mu-(1-\gamma)m]} e^{-p\sigma s} ds \right)^{1/p} \left(\int_0^t \varphi^q(s) v^{qm}(s)\, ds \right)^{1/q}, \qquad t > 0.$$

The last term in the previous relation is finite for each t fixed. This follows from the fact φ is continuous and u is in $C_{1-\gamma}[0,T]$.

Since $\lambda_1 - 1 = p[\mu - (1-\gamma)m]$, $\lambda_2 - 1 = p(\alpha-1)$ and $\lambda_1, \lambda_2,\ p\sigma > 0$, we may apply Lemma 2.5.1 (with ν replaced by λ_2, λ replaced by λ_1 and ω replaced by $p\sigma$) to get

$$\int_0^t (t-s)^{\alpha-1} s^{\mu-(1-\gamma)m} e^{-\sigma s} \varphi(s) v^m(s)\, ds \leq C_1 t^{\alpha-1} \left(\int_0^t \varphi^q(s) v^{qm}(s)\, ds \right)^{1/q}, \qquad (3.26)$$

where C_1 is the constant appearing in Lemma 2.5.1 corresponding to the present exponents. That is

$$C_1 = \left[2^{p(1-\alpha)} \Gamma(\lambda_1) \left(1 + \lambda_1 (\lambda_1 + 1) / \lambda_2\right) (p\sigma)^{-\lambda_1} \right]^{1/p}.$$

Combining (3.25) and (3.26) we entail that

$$v(t) \leq |b| + t^{-\beta(1-\alpha)} \hat{C}_1 \left(\int_0^t \varphi^q(s) \, v^{qm}(s) \, ds \right)^{1/q}, \qquad t > 0, \tag{3.27}$$

where $\hat{C}_1 = C_1/\Gamma(\alpha)$. Multiplying both sides of (3.27) by $t^{\beta(1-\alpha)}$ we obtain

$$t^{\beta(1-\alpha)} v(t) \leq |b| \, t^{\beta(1-\alpha)} + \hat{C}_1 \left(\int_0^t \varphi^q(s) \, v^{qm}(s) \, ds \right)^{1/q}, \qquad t > 0. \tag{3.28}$$

Let $z(t)$ denote the left-hand side of (3.28). Inserting the term $s^{-qm\beta(1-\alpha)} s^{qm\beta(1-\alpha)}$ inside the integral gives

$$z(t) \leq |b| \, t^{\beta(1-\alpha)} + \hat{C}_1 \left(\int_0^t \varphi^q(s) \, s^{-qm\beta(1-\alpha)} z^{qm}(s) \, ds \right)^{1/q}, \qquad t > 0, \tag{3.29}$$

and raising both sides of (3.29) to the power q, we get (using Lemma 2.5.2)

$$z^q(t) \leq 2^{q-1} \left(|b|^q \, t^{q\beta(1-\alpha)} + \hat{C}_1^q \int_0^t \varphi^q(s) \, s^{-qm\beta(1-\alpha)} z^{qm}(s) \, ds \right), \qquad t > 0. \tag{3.30}$$

Let us set

$$w(t) = \hat{C}_1^q \int_0^t \varphi^q(s) \, s^{-qm\beta(1-\alpha)} z^{qm}(s) \, ds, \qquad t > 0. \tag{3.31}$$

Then, by the continuity of z and the assumption $\varphi(t) \, t^{-m\beta(1-\alpha)} \in L^q(0, \infty)$ the integrand is summable and $w(0) = 0$, and by differentiation

$$w'(t) = \hat{C}_1^q \varphi^q(t) \, t^{-qm\beta(1-\alpha)} z^{qm}(t). \tag{3.32}$$

Moreover, it is clear that φ and v are nonnegative continuous functions in $\mathbb{R}^+$, and thus w is a continuous, nonnegative and nondecreasing function in $\mathbb{R}^+$.

Now, we would like to estimate the right hand side of (3.32) in terms of w.

From (3.30) and (3.31) we entail that

$$z^q(t) \leq 2^{q-1}\left(|b|^q t^{q\beta(1-\alpha)} + w(t)\right), \qquad t > 0.$$

Raising both sides to the power m and using Lemma 2.5.2, we get

$$z^{qm}(t) \leq 2^{mq-1}\left(|b|^{mq} t^{mq\beta(1-\alpha)} + w^m(t)\right). \tag{3.33}$$

Next, a substitution of (3.33) into (3.32) yields

$$w'(t) \leq 2^{mq-1}\hat{C}_1^q \varphi^q(t)\, t^{-qm\beta(1-\alpha)}\left(|b|^{mq} t^{mq\beta(1-\alpha)} + w^m(t)\right)$$

$$\leq 2^{mq-1}|b|^{mq}\hat{C}_1^q \varphi^q(t) + 2^{mq-1}\hat{C}_1^q t^{-qm\beta(1-\alpha)}\varphi^q(t)\, w^m(t), \qquad t > 0. \tag{3.34}$$

Applying Lemma 2.5.3 (with $w(u) = u^m$) we infer that

$$w(t) \leq G^{-1}\left[G(w(0) + l(t)) + k(t)\right],$$

where

$l(t) = 2^{mq-1}|b|^{mq}\hat{C}_1^q \int_0^t \varphi^q(s)\,ds$ and $k(t) = 2^{mq-1}\hat{C}_1^q \int_0^t s^{-qm\beta(1-\alpha)}\varphi^q(s)\,ds$. Since

$G(r) = \int_{r_0}^{r} \frac{ds}{s^m}$, $r > 0$, $r_0 > 0$, then

$$G(r) = \frac{r^{1-m}}{1-m} - \frac{r_0^{1-m}}{1-m}$$

and

$$G^{-1}(y) = \left[r_0^{1-m} - (m-1)\,y\right]^{-\frac{1}{m-1}}.$$

That is

$$w(t) \leq G^{-1}\left[\frac{l(t)^{1-m}}{1-m} - \frac{l(t_0)^{1-m}}{1-m} + k(t)\right]$$

$$\leq \left[l(t_0)^{1-m} - (m-1)\left(\frac{l(t)^{1-m}}{1-m} - \frac{l(t_0)^{1-m}}{1-m} + k(t)\right)\right]^{-\frac{1}{m-1}}$$

$$\leq \left[l(t)^{1-m} - (m-1)\,k(t)\right]^{-\frac{1}{m-1}}$$

as long as

$$l(t)^{m-1}\,k(t) < \frac{1}{m-1}.$$

In particular, if $\left(\int_0^t \varphi^q(s)\,ds\right)^{m-1}\left(\int_0^t s^{-qm\beta(1-\alpha)}\varphi^q(s)\,ds\right) < \mathbf{L}$ then $w(t) \leq K_1$ for some positive constant K_1, and thus from (3.29) we find that

$$z(t) \leq |b|\,t^{\beta(1-\alpha)} + K_1^{1/q}$$

or

$$t^{\beta(1-\alpha)}v(t) \leq |b|\,t^{\beta(1-\alpha)} + K_1^{1/q},$$

then

$$v(t) \leq |b| + K_1^{1/q} t^{-\beta(1-\alpha)} \leq C, \qquad t \geq t_0 > 0$$

for some positive constant C. This yields that $|u(t)| \leq C t^{\gamma-1}$ for $t \geq t_0 > 0$ and the proof is complete.

3.4 Non-existence of Solutions

In this section we consider the Cauchy problem of fractional order with a polynomial nonlinearity with variable coefficient

$$\begin{cases} \left(D_{0^+}^{\alpha,\beta} u\right)(t) \geq t^{\delta} |u(t)|^m, \quad t > 0,\ m > 1,\ \delta \in \mathbb{R} \\ \left(D_{0^+}^{\gamma-1} u\right)(0) = b > 0, \end{cases} \tag{3.35}$$

where $D_{0^+}^{\alpha,\beta}$ is the Hilfer Fractional Derivative (**HFD**) of order $0 < \alpha < 1$ and type $0 \leq \beta \leq 1$ and $\gamma = \alpha + \beta - \alpha\beta$.

There are several ways to investigate the non-existence and the blow up of solutions. We will follow the paper by Laskri and Tatar [63], the proof is based on the test function method and the integration by parts formula.

For this we need the following lemma.

Lemma 3.4.1: *If* $\alpha > 0$ and $f \in C[a,b]$, *then*

$$\left(I_{a+}^{\alpha} f\right)(a) = \lim_{t \to a} \left(I_{a+}^{\alpha} f\right)(t) = 0$$

and

$$\left(I_{b^-}^{\alpha} f\right)(b) = \lim_{t \to b} \left(I_{b^-}^{\alpha} f\right)(t) = 0.$$

Proof : Since $f \in C[a,b]$, then on $[a,b]$, we have

$$|f(t)| < M,$$

for some positive constant M.

Therefore

$$\left|\left(I_{a+}^{\alpha} f\right)(t)\right| \leq \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} |f(s)|\, ds$$

$$\leq \frac{M}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1}\, ds$$

$$\leq \frac{M}{\alpha\Gamma(\alpha)} \left[-(t-s)^{\alpha}\right]_{s=a}^{t} = \frac{M}{\alpha\Gamma(\alpha)} (t-a)^{\alpha}.$$

As $\alpha > 0$ we see that

$$\left(I_{a+}^{\alpha} f\right)(a) = \lim_{t \to a} \left(I_{a+}^{\alpha} f\right)(t) = 0.$$

Similarity

$$\left|\left(I_{b^-}^{\alpha} f\right)(t)\right| \leq \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} |f(s)|\, ds$$

$$\leq \frac{M}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1}\, ds$$

$$\leq \frac{M}{\alpha\Gamma(\alpha)} \left[(s-t)^{\alpha}\right]_{s=t}^{b} = \frac{M}{\alpha\Gamma(\alpha)} (b-t)^{\alpha}.$$

As $\alpha > 0$ we see that

$$\left(I_{b^-}^{\alpha} f\right)(b) = \lim_{t \to b} \left(I_{b^-}^{\alpha} f\right)(t) = 0.$$

Theorem (3.4.1): *Assume that* $\delta > -\alpha$ *and* $1 < m \leq \frac{\delta+1}{1-\alpha}$. *Then, Problem (3.35) does not admit global nontrivial solutions in* $C^{\gamma}_{1-\gamma}[0,T]$, *when* $b > 0$.

Proof : Assume, on the contrary, that a nontrivial solution u exists for all time $t > 0$. Let $\varphi \in C^1([0,\infty))$ be a test function satisfying : $\varphi(t) \geq 0$, φ is non-increasing such that

$$\varphi(t) := \begin{cases} 1, & t \in [0, T/2] \\ 0, & t \in [T, \infty) \end{cases}$$

for some $T > 0$. Multiplying the inequality in (3.35) by $\varphi(t)$ and integrating we get

$$\int_0^T \left(D^{\alpha,\beta}_{0^+} u\right)(t)\, \varphi(t)\, dt \geq \int_0^T t^{\delta} |u(t)|^m \varphi(t)\, dt$$

and from the definition of $\left(D^{\alpha,\beta}_{0^+} u\right)(t)$ we can write

$$\int_0^T I^{\beta(1-\alpha)}_{0^+} \frac{d}{dt} \left(I^{1-\gamma}_{0^+} u\right)(t)\, \varphi(t)\, dt \geq \int_0^T t^{\delta} |u(t)|^m \varphi(t)\, dt. \tag{3.36}$$

By virtue of Lemma 2.2.9, we may deduce from (3.36) that

$$\int_0^T \frac{d}{dt} \left(I^{1-\gamma}_{0^+} u\right)(t) \left(I^{\beta(1-\alpha)}_{T-} \varphi\right)(t)\, dt \geq \int_0^T t^{\delta} |u(t)|^m \varphi(t)\, dt. \tag{3.37}$$

An integration by parts in (3.37) yields

$$\left[\left(I^{1-\gamma}_{0^+} u\right)(t) \left(I^{\beta(1-\alpha)}_{T-} \varphi\right)(t)\right]_{t=0}^{T} - \int_0^T \left(I^{1-\gamma}_{0^+} u\right)(t) \frac{d}{dt} \left(I^{\beta(1-\alpha)}_{T-} \varphi\right)(t)\, dt$$

$$\geq \int_0^T t^\delta \left|u\left(t\right)\right|^m \varphi\left(t\right) dt.$$

Note that the derivative inside the second term in the right hand side is meaningful because $\varphi \in [0, \infty)$ and $\varphi\left(T\right) = 0$ so the Riemann-Liouville and Caputo exist and are equal.

By using Lemma 3.4.1 we see that $\left(I_{T-}^{\beta(1-\alpha)}\varphi\right)\left(T\right) = 0$ and also from initial condition $\left(I_{0^+}^{1-\gamma}u\right)\left(0\right) = \left(D_{0^+}^{\gamma-1}u\right)\left(0\right) = b$, so

$$-b\left(I_{T-}^{\beta(1-\alpha)}\varphi\right)\left(0\right) - \int_0^T \left(I_{0^+}^{1-\gamma}u\right)\left(t\right) \frac{d}{dt}\left(I_{T-}^{\beta(1-\alpha)}\varphi\right)\left(t\right) dt \geq \int_0^T t^\delta \left|u\left(t\right)\right|^m \varphi\left(t\right) dt, \tag{3.38}$$

It appears from Definition 2.2.4 that

$$-b\left(I_{T-}^{\beta(1-\alpha)}\varphi\right)\left(0\right) + \int_0^T \left(I_{0^+}^{1-\gamma}u\right)\left(t\right)\left(D_{T-}^{1-\beta(1-\alpha)}\varphi\right)\left(t\right) dt \geq \int_0^T t^\delta \left|u\left(t\right)\right|^m \varphi\left(t\right) dt \tag{3.39}$$

and from Lemma 2.2.5 we see that

$$\begin{aligned} -b\left(I_{T-}^{\beta(1-\alpha)}\varphi\right)\left(0\right) + \int_0^T \left(I_{0^+}^{1-\gamma}u\right)\left(t\right)&\left[\frac{1}{\Gamma\left[\beta\left(1-\alpha\right)\right]}\left(\frac{\varphi\left(T\right)}{\left(T-t\right)^{1-\beta(1-\alpha)}}\right.\right. \\ &\left.\left. - \int_t^T \frac{\varphi'\left(s\right) ds}{\left(s-t\right)^{1-\beta(1-\alpha)}}\right)\right] \geq \int_0^T t^\delta \left|u\left(t\right)\right|^m \varphi\left(t\right) dt. \end{aligned} \tag{3.40}$$

Since $\varphi\left(T\right) = 0$ the relation (3.40) becomes

$$-b\left(I_{T-}^{\beta(1-\alpha)}\varphi\right)\left(0\right) - \int_0^T \left(I_{0^+}^{1-\gamma}u\right)\left(t\right)\left(I_{T-}^{\beta(1-\alpha)}\varphi'\right)\left(t\right) dt \geq \int_0^T t^\delta \left|u\left(t\right)\right|^m \varphi\left(t\right) dt.$$

Lemma 2.2.9 again allows us to write

$$-b\left(I_{T-}^{\beta(1-\alpha)}\varphi\right)(0)-\int_0^T\varphi'(t)\left(I_{0^+}^{\beta(1-\alpha)}I_{0^+}^{1-\gamma}u\right)(t)\,dt\geq\int_0^T t^{\delta}\left|u(t)\right|^m\varphi(t)\,dt$$

and by Lemma 2.2.6

$$-b\left(I_{T-}^{\beta(1-\alpha)}\varphi\right)(0)-\int_0^T\varphi'(t)\left(I_{0^+}^{1-\alpha}u\right)(t)\,dt\geq\int_0^T t^{\delta}\left|u(t)\right|^m\varphi(t)\,dt. \tag{3.41}$$

Notice that

$$\int_0^T\varphi'(t)\left(I_{0^+}^{1-\alpha}u\right)(t)\,dt=\frac{1}{\Gamma(1-\alpha)}\int_0^T\varphi'(t)\int_0^t\frac{u(s)}{(t-s)^{\alpha}}dsdt$$

$$\leq\frac{1}{\Gamma(1-\alpha)}\int_0^T\left|\varphi'(t)\right|\int_0^t\frac{|u(s)|}{(t-s)^{\alpha}}dsdt.$$

Since $\varphi(t)$ is nonincreasing, $\varphi(s)\geq\varphi(t)$ for all $t\geq s$, and thus

$$\frac{1}{\varphi(s)^{1/m}}\leq\frac{1}{\varphi(t)^{1/m}},\ 0\leq s\leq t<T,\ m>1.$$

Also we have

$$\varphi'(t)=0,\ t\in[0,T/2].$$

Therefore

$$\begin{aligned}
\int_0^T \varphi'(t)\left(I_{0^+}^{1-\alpha}u\right)(t)\,dt &\leq \tfrac{1}{\Gamma(1-\alpha)}\int_0^T\left|\varphi'(t)\right|\int_0^t \tfrac{|u(s)|}{(t-s)^\alpha}\tfrac{\varphi(s)^{1/m}}{\varphi(s)^{1/m}}dsdt\\
&\leq \tfrac{1}{\Gamma(1-\alpha)}\int_0^T \tfrac{\left|\varphi'(t)\right|}{\varphi(t)^{1/m}}\int_0^t \tfrac{|u(s)|}{(t-s)^\alpha}\varphi(s)^{1/m}\,dsdt\\
&\leq \tfrac{1}{\Gamma(1-\alpha)}\int_{T/2}^T \tfrac{\left|\varphi'(t)\right|}{\varphi(t)^{1/m}}\int_0^t \tfrac{|u(s)|}{(t-s)^\alpha}\varphi(s)^{1/m}\,dsdt.
\end{aligned}$$

From Definition 2.2.1 we have

$$\int_0^T \varphi'(t)\left(I_{0^+}^{1-\alpha}u\right)(t)\,dt \leq \int_{T/2}^T \frac{\left|\varphi'(t)\right|}{\varphi(t)^{1/m}}\left(I_{0^+}^{1-\alpha}\left[\varphi^{1/m}|u|\right]\right)(t)\,dt$$

and by Lemma 2.2.9, we find

$$\int_0^T \varphi'(t)\left(I_{0^+}^{1-\alpha}u\right)(t)\,dt \leq \int_{T/2}^T \left(I_{T-}^{1-\alpha}\frac{\left|\varphi'\right|}{\varphi^{1/m}}\right)(t)\,\varphi(t)^{1/m}|u(t)|\,dt.$$

(Note that we may assume that $\left|\varphi'(t)\right|\varphi(t)^{-1/m}$ is summable even though $\varphi(t)\to 0$ as $t\to T$, for otherwise we consider $\varphi^\lambda(t)$ with sufficiently large exponent λ).

Next, we multiply by $t^{\delta/m}.t^{-\delta/m}$ inside the integral in the right hand side

$$\int_0^T \varphi'(t)\left(I_{0^+}^{1-\alpha}u\right)(t)\,dt \leq \int_{T/2}^T \left(I_{T-}^{1-\alpha}\frac{\left|\varphi'\right|}{\varphi^{1/m}}\right)(t)\,\varphi(t)^{1/m}\frac{t^{\delta/m}}{t^{\delta/m}}|u(t)|\,dt.$$

For $-\alpha<\delta<0$ we have $t^{-\delta/m}<T^{-\delta/m}$ (because $t<T$) and for $\delta>0$ we get $t^{-\delta/m}<2^{\delta/m}T^{-\delta/m}$ (because $T/2<t$), that is

$$t^{-\delta/m}<\max\left\{1,2^{\delta/m}\right\}T^{-\delta/m}.$$

Therefore

$$\int_0^T \varphi'(t)\left(I_{0^+}^{1-\alpha}u\right)(t)\,dt$$

$$\leq \max\left\{1, 2^{\delta/m}\right\} T^{-\delta/m} \int_{T/2}^T \left(I_{T-}^{1-\alpha}\frac{\left|\varphi'\right|}{\varphi^{1/m}}\right)(t)\, t^{\delta/m}\varphi(t)^{1/m}\left|u(t)\right| dt. \tag{3.42}$$

A simple application of the Young inequality (Theorem 2.5.2) with m and m' such that $\frac{1}{m}+\frac{1}{m'}=1$ gives

$$\int_0^T \varphi'(t)\left(I_{0^+}^{1-\alpha}u\right)(t)\,dt$$

$$\leq \frac{1}{m}\int_{T/2}^T t^\delta\varphi(t)\left|u(t)\right|^m dt + \frac{\left(\max\left\{1,2^{\delta/m}\right\}\right)^{m'}}{m'}T^{-\frac{\delta m'}{m}}\int_{T/2}^T\left(I_{T-}^{1-\alpha}\frac{\left|\varphi'\right|}{\varphi^{1/m}}\right)^{m'}(t)\,dt$$

$$\leq \frac{1}{m}\int_0^T t^\delta\varphi(t)\left|u(t)\right|^m dt + \frac{\left(\max\left\{1,2^{\delta/m}\right\}\right)^{m'}}{m'}T^{-\frac{\delta m'}{m}}\int_{T/2}^T\left(I_{T-}^{1-\alpha}\frac{\left|\varphi'\right|}{\varphi^{1/m}}\right)^{m'}(t)\,dt. \tag{3.43}$$

Clearly from (3.41) and (3.43) we see that

$$-b\left(I_{T-}^{\beta(1-\alpha)}\varphi\right)(0) + \frac{\left(\max\left\{1,2^{\delta/m}\right\}\right)^{m'}}{m'}T^{-\frac{\delta m'}{m}}\int_{T/2}^T\left(I_{T-}^{1-\alpha}\frac{\left|\varphi'\right|}{\varphi^{1/m}}\right)^{m'}(t)\,dt$$

$$\geq \left(1-\frac{1}{m}\right)\int_0^T t^\delta\left|u(t)\right|^m\varphi(t)\,dt,$$

or since $b>0$

$$\frac{1}{m'}\int_0^T t^\delta\left|u(t)\right|^m\varphi(t)\,dt \leq \frac{\left(\max\left\{1,2^{\delta/m}\right\}\right)^{m'}}{m'}T^{-\frac{\delta m'}{m}}\int_{T/2}^T\left(I_{T-}^{1-\alpha}\frac{\left|\varphi'\right|}{\varphi^{1/m}}\right)^{m'}(t)\,dt. \tag{3.44}$$

Therefore, by Definition 2.2.2 we have

$$\int_0^T t^\delta |u(t)|^m \varphi(t)\, dt$$

$$\leq \left(\max\left\{1, 2^{\delta/m}\right\}\right)^{m'} T^{-\frac{\delta m'}{m}} \int_{T/2}^T \left(\frac{1}{\Gamma(1-\alpha)} \int_t^T (s-t)^{-\alpha} \frac{\left|\varphi'(s)\right|}{\varphi(s)^{1/m}} ds\right)^{m'} dt. \quad (3.45)$$

The change of variable $\sigma T = t$ yields

$$\int_0^T t^\delta |u(t)|^m \varphi(t)\, dt$$

$$\leq \left(\max\left\{1, 2^{\delta/m}\right\}\right)^{m'} T^{-\frac{\delta m'}{m}} \int_{1/2}^1 \left(\frac{1}{\Gamma(1-\alpha)} \int_{\sigma T}^T (s-\sigma T)^{-\alpha} \frac{\left|\varphi'(s)\right|}{\varphi(s)^{1/m}} ds\right)^{m'} T d\sigma. \quad (3.46)$$

Another change of variable $s = rT$ gives

$$\int_0^T t^\delta |u(t)|^m \varphi(t)\, dt$$

$$\leq \left(\max\left\{1, 2^{\delta/m}\right\}\right)^{m'} T^{-\frac{\delta m'}{m}} \int_{1/2}^1 \left(\frac{1}{\Gamma(1-\alpha)} \int_\sigma^1 (rT-\sigma T)^{-\alpha} \frac{\left|\varphi'(r)\right|}{\varphi(r)^{1/m}} dr\right)^{m'} T d\sigma,$$

or

$$\int_0^T t^\delta |u(t)|^m \varphi(t)\, dt$$

$$\leq \frac{\left(\max\left\{1, 2^{\delta/m}\right\}\right)^{m'}}{\Gamma^{m'}(1-\alpha)} T^{1-\alpha m'-\delta m'/m} \int_{1/2}^1 \left(\int_\sigma^1 (r-\sigma)^{-\alpha} \frac{\left|\varphi'(r)\right|}{\varphi(r)^{1/m}} dr\right)^{m'} d\sigma. \quad (3.47)$$

At this point it is clear that we may assume that the integral term in the right hand side of (3.47) is bounded, that is

$$\int_{1/2}^{1}\left(\int_{\sigma}^{1}(r-\sigma)^{-\alpha}\frac{\left|\varphi^{'}(r)\right|}{\varphi(r)^{1/m}}dr\right)^{m^{'}}d\sigma\leq K_{1},$$

for some positive constant K_1, for otherwise we consider $\varphi^{\lambda}(r)$ with some sufficiently large λ. Therefore

$$\int_{0}^{T}t^{\delta}\left|u(t)\right|^{m}\varphi(t)\,dt\leq K_{2}T^{1-\alpha m^{'}-\delta m^{'}/m}, \tag{3.48}$$

with

$$K_{2}:=\frac{\left(\max\left\{1,2^{\delta/m}\right\}\right)^{m^{'}}}{\Gamma^{m^{'}}(1-\alpha)}K_{1}.$$

If $m<\frac{\delta+1}{1-\alpha}$ we see that $1-\alpha m^{'}-\delta m^{'}/m<0$ and consequently $T^{1-\alpha m^{'}-\delta m^{'}/m}\rightarrow 0$ as $T\rightarrow\infty$. Then from (3.48) we obtain

$$\lim_{T\rightarrow\infty}\int_{0}^{T}t^{\delta}\left|u(t)\right|^{m}\varphi(t)\,dt=0.$$

We reach a contradiction since the solution is supposed to be nontrivial.

In the case $m=\frac{\delta+1}{1-\alpha}$ we have $1-\alpha m^{'}-\delta m^{'}/m=0$ and the relation (3.48) ensures that

$$\lim_{T\rightarrow\infty}\int_{0}^{T}t^{\delta}\left|u(t)\right|^{m}\varphi(t)\,dt\leq K_{2}. \tag{3.49}$$

Moreover, it is clear that

$$\int_{T/2}^{T} \left(I_{T-}^{1-\alpha} \frac{|\varphi'|}{\varphi^{1/m}} \right)(t)\, t^{\delta/m} \varphi(t)^{1/m} |u(t)|\, dt$$

$$\leq \left[\int_{T/2}^{T} \left(I_{T-}^{1-\alpha} \frac{|\varphi'|}{\varphi^{1/m}} \right)^{m'} (t)\, dt \right]^{\frac{1}{m'}} \left[\int_{T/2}^{T} t^{\delta} \varphi(t) |u(t)|^{m}\, dt \right]^{\frac{1}{m}}.$$

This relation, together with (3.41) and (3.42), implies that

$$\int_{0}^{T} t^{\delta} \varphi(t) |u(t)|^{m}\, dt \leq K_3 \left[\int_{T/2}^{T} t^{\delta} \varphi(t) |u(t)|^{m}\, dt \right]^{\frac{1}{m}}$$

for some positive constant K_3, with

$$\lim_{T\to\infty} \int_{T/2}^{T} t^{\delta} \varphi(t) |u(t)|^{m}\, dt = 0$$

due to the convergence of the integral in (3.49). This leads again to a contradiction.

This completes the proof of Theorem 3.4.1.

Chapter 4

Fractional Differential Problems with Hilfer-Hadamard Fractional Derivative

In this chapter we discuss the existence, uniqueness and solution, the stability of solutions of the Cauchy type problem (4.2).

Motivated by the definition of Hilfer fractional derivative which interpolates the Riemann-Liouville fractional derivative and the Caputo fractional derivative, we introduce next a similar one which interpolates the Hadamard fractional derivative (Definition 2.3.3) and its corresponding Hadamard-Caputo fractional derivative (Definition 2.4.2).

Definition (4.1): *(Hilfer–Hadamard Fractional Derivative (HHFD)) The left sided fractional derivative of order* α $(0 < \alpha < 1)$ *and type* $0 \leq \beta \leq 1$ *with respect to* x *is defined by*

$$\left(\mathcal{D}_{a^+}^{\alpha,\beta} f\right)(x) = \left(\mathcal{J}_{a^+}^{\beta(1-\alpha)} \mathcal{D}_{a^+}^{\alpha+\beta-\alpha\beta} f\right)(x) \tag{4.1}$$

where $\mathcal{D}_{a^+}^{\alpha+\beta-\alpha\beta}$ is the Hadamard fractional derivative, *for functions for which the expression on the right hand side exists.*

Indeed for $\beta = 0$ this derivative (4.1) reduces to the Hadamard fractional derivative (Definition 2.3.3) and when $\beta = 1$ we recover the Hadamard-Caputo fractional derivative Definition (2.4.2).

We will study the existence and uniqueness of a solution of the Cauchy type problem

$$\begin{cases} \left(\mathcal{D}_{a^+}^{\alpha,\beta} y\right)(x) = f(x, y), \quad x > a > 0 \\ \left(\mathcal{J}_{a^+}^{(1-\beta)(1-\alpha)} y\right)(a) = c, \end{cases} \tag{4.2}$$

where $0 < \alpha < 1$, $0 \leq \beta \leq 1$ and $\mathcal{D}_{a^+}^{\alpha,\beta}$ is the **HHFD** (4.1) of order α and type β and c is a real number. We consider the underlying spaces $C_{\delta;1-\gamma,\mu}^{\alpha,\beta}[a,b]$ defined by

$$C_{\delta;1-\gamma,\mu}^{\alpha,\beta}[a,b] = \left\{ y \in C_{1-\gamma,\log}[a,b], \mathcal{D}_{a^+}^{\alpha,\beta} y \in C_{\mu,\log}[a,b] \right\} \tag{4.3}$$

and

$$C_{1-\gamma,\log}^{\gamma}[a,b] = \left\{ y \in C_{1-\gamma,\log}[a,b], \mathcal{D}_{a+}^{\gamma} y \in C_{1-\gamma,\log}[a,b] \right\} \tag{4.4}$$

where $\gamma = \alpha + \beta - \alpha\beta$ and $0 \leq \mu < 1$. It is clear that $0 < \gamma < 1$ for $0 < \alpha, \beta < 1$. Here $C_{1-\gamma,\log}[a,b]$ and $C_{\mu,\log}[a,b]$ are weighted spaces of continuous functions on $(a,b]$ defined by

$$C_{\gamma,\log}[a,b] = \left\{ g : (a,b] \to \mathbb{R}\colon \left(\log \frac{x}{a}\right)^{\gamma} g(x) \in C[a,b] \right\}, \tag{4.5}$$

Our investigations are based on reducing the fractional differential problem to a Volterra integral equation of the second kind

$$y(x) = \frac{c}{\Gamma(\gamma)} \left(\log \frac{x}{a}\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_a^x \left(\log \frac{x}{t}\right)^{\alpha-1} f[t, y(t)] \frac{dt}{t}, \quad x > a, \tag{4.6}$$

and on using the Banach fixed point theorem.

4.1 Equivalence of the Cauchy Type Problem and the Volterra Integral Equation

In this section we prove the equivalence of the Cauchy type problem (4.2) and the nonlinear Volterra integral equation (4.6) in the sense that, if $y \in C^{\gamma}_{1-\gamma,\log}[a,b]$ satisfies one of them, then it also satisfies the other one. To establish this result, we assume that the function $f[x, y(x)]$ belongs to $C_{\mu,\log}[a,b]$ for any $y \in \mathbf{G} \subset \mathbb{R}$. We will need the following lemma.

Lemma 4.1.1: *Let* $0 < a < b < \infty$, $\alpha > 0$, $0 \leq \mu < 1$ *and* $g \in C_{\mu,\log}[a,b]$. *If* $\alpha > \mu$, *then* $\mathcal{J}^{\alpha}_{a+}g$ *is continuous on* $[a,b]$ *and*

$$\mathcal{J}^{\alpha}_{a+}g(a) = \lim_{x \to a} \mathcal{J}^{\alpha}_{a+}g(x) = 0.$$

Proof : Since $g \in C_{\mu,\log}[a,b]$ then $\left(\log \frac{x}{a}\right)^{\mu} g(x)$ is continuous on $[a,b]$ and on $[a,b]$ we have

$$\left|\left(\log \frac{x}{a}\right)^{\mu} g(x)\right| \leq M,$$

for some positive constant M. Therefore

$$\left|\left(\mathcal{J}^{\alpha}_{a+}g\right)(x)\right| \leq M \left(\mathcal{J}^{\alpha}_{a+}\left(\log \frac{x}{a}\right)^{-\mu}\right)(x)$$

and by using Property 2.3.1 (with $\beta = 1 - \mu > 0$) we have

$$\left|\left(\mathcal{J}^{\alpha}_{a+}g\right)(x)\right| \leq M \frac{\Gamma(1-\mu)}{\Gamma(\alpha+1-\mu)} \left(\log \frac{x}{a}\right)^{\alpha-\mu}.$$

As $\alpha > \mu$, we obtain the result.

Theorem 4.1.1: *Let $\gamma = \alpha + \beta - \alpha\beta$ where $0 < \alpha < 1$ and $0 < \beta < 1$. Let $f : (a, b] \times \mathbb{R} \to \mathbb{R}$ $(a > 0)$ be a function such that $f(x, y) \in C_{\mu,\log}[a, b]$ for any $y \in C_{\mu,\log}[a, b]$ with $1 - \gamma \leq \mu < 1 - \beta(1 - \alpha)$.*

If $y \in C^{\gamma}_{1-\gamma,\log}[a, b]$, then y satisfies the (CFDP) (4.2) if and only if y satisfies the (IE) (4.6).

Proof : First we prove the **necessity.** Let $y \in C^{\gamma}_{1-\gamma,\log}[a, b]$ be a solution of problem (4.2). We want to prove that y is also a solution of the integral equation (4.6). By the definition of the space $C^{\gamma}_{1-\gamma,\log}[a, b]$ relation (4.4) above, we have

$$\delta\left(\mathcal{J}^{1-\gamma}_{a^+} y\right) = \mathcal{D}^{\gamma}_{a^+} y \in C_{1-\gamma,\log}[a, b].$$

Moreover, by Lemma 2.3.2(b) we have $\mathcal{J}^{1-\gamma}_{a^+} y \in C[a, b]$ since $y \in C_{1-\gamma,\log}[a, b]$. Then by Definition 2.1.7 we have

$$\left(\mathcal{J}^{1-\gamma}_{a^+} y\right)(x) \in C^{1}_{\delta,1-\gamma}[a, b].$$

Thus we can apply Theorem 2.3.1 (with f replaced by y) to get

$$\left(\mathcal{J}^{\gamma}_{a^+} \mathcal{D}^{\gamma}_{a^+} y\right)(x) = y(x) - \frac{\left(\mathcal{J}^{1-\gamma}_{a^+} y\right)(a)}{\Gamma(\gamma)} \left(\log \frac{x}{a}\right)^{\gamma-1}, \quad x \in (a, b] \tag{4.7}$$

or

$$\left(\mathcal{J}^{\gamma}_{a^+} \mathcal{D}^{\gamma}_{a^+} y\right)(x) = y(x) - \frac{c}{\Gamma(\gamma)} \left(\log \frac{x}{a}\right)^{\gamma-1}, \quad x \in (a, b] \tag{4.8}$$

where c comes from the initial condition in (4.2). By our hypothesis $f[x, y(x)] \in C_{\mu,\log}[a,b]$, since $y \in C_{1-\gamma,\log}[a,b] \subset C_{\mu,\log}[a,b]$, and Lemma 2.3.2 (a) and 2.3.2 (b) we see that the integral $\mathcal{J}_{a+}^{\alpha} f[x, y(x)] \in C_{\mu-\alpha,\log}[a,b]$ for $\mu > \alpha$ and $\mathcal{J}_{a+}^{\alpha} f[x, y(x)] \in C[a,b]$ for $\mu \leq \alpha$. Applying the operator $\mathcal{J}_{a+}^{\alpha}$ to both sides of (4.2) we get

$$\mathcal{J}_{a+}^{\alpha} \mathcal{J}_{a+}^{\beta(1-\alpha)} \left(\mathcal{D}_{a+}^{\gamma} y\right)(x) = \mathcal{J}_{a+}^{\alpha} f[x, y(x)], \qquad x \in (a,b].$$

We can sum up the exponents by Property 2.3.2 to get

$$\mathcal{J}_{a+}^{\alpha+\beta(1-\alpha)} \mathcal{D}_{a+}^{\gamma} y(x) = \mathcal{J}_{a+}^{\alpha} f[x, y(x)], \qquad x \in (a,b]$$

or

$$\left(\mathcal{J}_{a+}^{\gamma} \mathcal{D}_{a+}^{\gamma} y\right)(x) = \left(\mathcal{J}_{a+}^{\alpha} f[t, y(t)]\right)(x), \qquad x \in (a,b]. \tag{4.9}$$

From (4.8) and (4.9) we obtain

$$y(x) = \frac{c}{\Gamma(\gamma)} \left(\log \frac{x}{a}\right)^{\gamma-1} + \left(\mathcal{J}_{a+}^{\alpha} f[t, y(t)]\right)(x),$$

which is the equation (4.6), and hence the necessity is proved.

Now we prove the sufficiency. Let $y \in C_{1-\gamma,\log}^{\gamma}[a,b]$ satisfy the equation (4.6), then $\mathcal{D}_{a+}^{\gamma} y$ exists and $\mathcal{D}_{a+}^{\gamma} y \in C_{1-\gamma,\log}[a,b]$. Applying the operator $\mathcal{D}_{a+}^{\gamma}$ to both sides of (4.6) we find

$$\left(\mathcal{D}_{a+}^{\gamma} y\right)(x) = \frac{c}{\Gamma(\gamma)} \mathcal{D}_{a+}^{\gamma} \left(\log \frac{t}{a}\right)^{\gamma-1}(x) + \left(\mathcal{D}_{a+}^{\gamma} \mathcal{J}_{a+}^{\alpha} f[t, y(t)]\right)(x).$$

By using Lemma 2.3.1, Definition 2.3.3, Property 2.3.2 and the hypothesis $f(x,y) \in C_{\mu,\log}[a,b]$, we have

$$\begin{aligned}\left(\mathcal{D}_{a+}^{\gamma}y\right)(x) &= \delta\left(\mathcal{J}_{a+}^{1-\gamma}\mathcal{J}_{a+}^{\alpha}f[t,y(t)]\right)(x)\\ &= \delta\left(\mathcal{J}_{a^+}^{1-\beta(1-\alpha)}f[t,y(t)]\right)(x)\\ &= \left(\mathcal{D}_{a^+}^{\beta(1-\alpha)}f[t,y(t)]\right)(x), \qquad x\in(a,b].\end{aligned} \tag{4.10}$$

From (4.10) and the fact that $\mathcal{D}_{a+}^{\gamma}y \in C_{1-\gamma,\log}[a,b]$, we obtain that

$$\mathcal{D}_{a^+}^{\beta(1-\alpha)}f[x,y(x)] \in C_{1-\gamma,\log}[a,b].$$

Next, applying the operator $\mathcal{J}_{a^+}^{\beta(1-\alpha)}$ to both sides of (4.10) we get

$$\left(\mathcal{J}_{a^+}^{\beta(1-\alpha)}\mathcal{D}_{a+}^{\gamma}y\right)(x) = \left(\mathcal{J}_{a^+}^{\beta(1-\alpha)}\mathcal{D}_{a^+}^{\beta(1-\alpha)}f[t,y(t)]\right)(x).$$

That is

$$\mathcal{J}_{a^+}^{\beta(1-\alpha)}\delta\left(\mathcal{J}_{a^+}^{1-\gamma}y\right)(x) = \left(\mathcal{J}_{a^+}^{\beta(1-\alpha)}\mathcal{D}_{a^+}^{\beta(1-\alpha)}f[t,y(t)]\right)(x).$$

By virtue of

$$\delta\left(\mathcal{J}_{a^+}^{1-\beta(1-\alpha)}f[t,y(t)]\right)(x) = \left(\mathcal{D}_{a^+}^{\beta(1-\alpha)}f[t,y(t)]\right)(x) \in C_{1-\gamma,\log}[a,b],$$

and $\gamma > \beta(1-\alpha)$ and Definition 2.1.7 we have $\mathcal{J}_{a^+}^{1-\beta(1-\alpha)}f \in C_{\delta;1-\gamma}^{1}[a,b]$ (see the first part of the proof for the continuity of $\mathcal{J}_{a^+}^{1-\beta(1-\alpha)}f$ for $\mu < 1-\beta(1-\alpha)$). Then

Theorem 2.3.1 allows us to write

$$\left(\mathcal{D}_{a^+}^{\alpha,\beta} y\right)(x) = f(x,y) - \frac{\left(\mathcal{J}_{a^+}^{1-\beta(1-\alpha)} f\right)(a)}{\Gamma\left[\beta(1-\alpha)\right]} \left(\log\frac{x}{a}\right)^{\beta(1-\alpha)-1}, \qquad x \in (a,b]. \quad (4.11)$$

Lemma 4.1.1 implies that

$$\left(\mathcal{J}_{a^+}^{1-\beta(1-\alpha)} f\right)(a) = 0$$

because $1-\beta(1-\alpha) > \mu$. Hence the relation (4.11) reduces to

$$\left(\mathcal{D}_{a^+}^{\alpha,\beta} y\right)(x) = f(x,y), \qquad x > a.$$

Now we show that the initial condition in (4.2) also holds. To this end we apply the operator $\mathcal{J}_{a^+}^{1-\gamma}$ to both sides of (4.6)

$$\left(\mathcal{J}_{a^+}^{1-\gamma} y\right)(x) = \frac{c}{\Gamma(\gamma)} \mathcal{J}_{a^+}^{1-\gamma} \left(\log\frac{t}{a}\right)^{\gamma-1}(x) + \left(\mathcal{J}_{a^+}^{1-\gamma} \mathcal{J}_{a^+}^{\alpha} f\left[t, y(t)\right]\right)(x)$$

and use the Property 2.3.1 (with α replaced by $1-\gamma$ and β by γ) and the Property 2.3.2 to obtain

$$\left(\mathcal{J}_{a^+}^{1-\gamma} y\right)(x) = c + \left(\mathcal{J}_{a^+}^{1-\beta(1-\alpha)} f\left[t, y(t)\right]\right)(x). \quad (4.12)$$

In (4.12), taking the limit as $x \to a$, we obtain

$$\left(\mathcal{J}_{a^+}^{1-\gamma} y\right)(a) = c + \left(\mathcal{J}_{a^+}^{1-\beta(1-\alpha)} f\left[t, y(t)\right]\right)(a).$$

But as mentioned above $\left(\mathcal{J}_{a^+}^{1-\beta(1-\alpha)} f\left[t, y\left(t\right)\right]\right)(a) = 0$, therefore

$$\left(\mathcal{J}_{a^+}^{1-\gamma} y\right)(a) = c$$

and the sufficiency is proved, which completes the proof of Theorem 4.1.1.

4.2 Existence and Uniqueness of a Solution

In this section we establish the existence of a unique solution to the Cauchy type problem (4.2) in the space $C_{\delta;1-\gamma,\mu}^{\alpha,\beta}[a,b]$ defined in (4.3) above under the conditions of Theorem 4.1.1 and an additional *Lipschitz* condition (Definition 2.5.1).

Theorem 4.2.1: *Let $\gamma = \alpha + \beta - \alpha\beta$ where $(0 < \alpha < 1,\ 0 \leq \beta \leq 1)$. Assume that $f : (a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ $(a > 0)$ is a function such that $f[x, y] \in C_{\mu,\log}[a, b]$ for any $y \in C_{\mu,\log}[a, b]$ with $1 - \gamma \leq \mu < 1 - \beta(1 - \alpha)$ and the Lipschitz condition (Definition 2.5.1) holds with respect to the second variable. Then there exists a unique solution y for the Cauchy type problem (4.2) in the space $C_{\delta;1-\gamma,\mu}^{\alpha,\beta}[a,b]$.*

Proof : First we prove the existence of a unique solution y in the space $C_{1-\gamma,\log}[a,b]$. According to Theorem 4.1.1, it suffices to prove the existence of a unique solution $y \in C_{1-\gamma,\log}[a,b]$ to the nonlinear Volterra integral equation (4.6).

Let us select x_1 in (a, b) such that

$$w_1 := \frac{A\Gamma(\gamma)}{\Gamma(\alpha+\gamma)} \left(\log \frac{x_1}{a}\right)^{\alpha} < 1, \tag{4.13}$$

where $A > 0$ is the Lipschitz constant in Definition 2.5.1. We start by proving that a unique solution $y \in C_{1-\gamma,\log}[a, x_1]$ to equation (4.6) exists on the interval $(a, x_1]$. It is easy to see that the space $C_{1-\gamma,\log}[a, x_1]$ is a complete metric space when equipped with the distance given by

$$d(y_1, y_2) = \|y_1 - y_2\|_{C_{1-\gamma,\log}[a,x_1]} := \max_{x\in[a,x_1]} \left| \left(\log \frac{x}{a}\right)^{1-\gamma} [y_1(x) - y_2(x)] \right|. \qquad (4.14)$$

The integral equation (4.6) takes the form

$$y(x) = (Ty)(x) \qquad (4.15)$$

where

$$(Ty)(x) = y_0(x) + \left(\mathcal{J}^{\alpha}_{a^+} f[t, y(t)]\right)(x) \qquad (4.16)$$

with

$$y_0(x) = \frac{c}{\Gamma(\gamma)} \left(\log \frac{x}{a}\right)^{\gamma-1}. \qquad (4.17)$$

We claim that T maps $C_{1-\gamma,\log}[a, x_1]$ to itself. Indeed, $y_0(x)$ given by (4.17) is clearly in $C_{1-\gamma,\log}[a, x_1]$. Also since $f[x, y(x)] \in C_{\mu,\log}[a, b]$ for any $y \in C_{\mu,\log}[a, b]$ with $\mu \in \mathbb{R}$ $(0 \leq \mu < 1)$, then by Lemma 2.3.2 (a) and 2.3.2 (b) the integral in the right–hand side of (4.16) belongs to $C_{\mu-\alpha,\log}[a, b]$ for $\mu > \alpha$ and to $C[a, b]$ for $\mu \leq \alpha$. Since $\mu - \alpha < 1 - \gamma$, $(\mu < 1 - \beta(1 - \alpha))$, and $1 - \gamma \geq 0$. Then, by Property 2.1.1 the right-hand side of (4.16) belongs to $C_{1-\gamma,\log}[a, b]$. Therefore, the operator T defined by (4.16) maps $C_{1-\gamma,\log}[a, x_1]$ onto $C_{1-\gamma,\log}[a, x_1]$, that is $Ty \in C_{1-\gamma,\log}[a, x_1]$.

Our second claim is that T is a contraction, that is

$$\|Ty_1 - Ty_2\|_{C_{1-\gamma,\log}[a,x_1]} \leq w_1 \|y_1 - y_2\|_{C_{1-\gamma,\log}[a,x_1]}, \ 0 < w_1 < 1. \tag{4.18}$$

This follows from (4.16), Definition 2.5.1, Lemma 2.3.2 (a) and the fact that

$$\begin{aligned}
\|Ty_1 - Ty_2\|_{C_{1-\gamma,\log}[a,x_1]} &= \left\|\mathcal{J}_{a+}^{\alpha} f\left[t, y_1(t)\right] - \mathcal{J}_{a+}^{\alpha} f\left[t, y_2(t)\right]\right\|_{C_{1-\gamma,\log}[a,x_1]} \\
&\leq \left\|\mathcal{J}_{a+}^{\alpha} \left|f[t, y_1(t)] - f[t, y_2(t)]\right|\right\|_{C_{1-\gamma,\log}[a,x_1]} \\
&\leq \left(\log \tfrac{x_1}{a}\right)^{\alpha} \tfrac{\Gamma(\gamma)}{\Gamma(\alpha+\gamma)} \|f[t, y_1(t)] - f[t, y_2(t)]\|_{C_{1-\gamma,\log}[a,x_1]} \\
&\leq A\left(\log \tfrac{x_1}{a}\right)^{\alpha} \tfrac{\Gamma(\gamma)}{\Gamma(\alpha+\gamma)} \|y_1(t) - y_2(t)\|_{C_{1-\gamma,\log}[a,x_1]} \\
&\leq w_1 \|y_1(t) - y_2(t)]\|_{C_{1-\gamma,\log}[a,x_1]}.
\end{aligned}$$

Our assumption (4.13) allows us to apply the Banach fixed point theorem (Theorem 2.5.1) to obtain a unique solution $y^* \in C_{1-\gamma,\log}[a, x_1]$ to the equation (4.6) on the interval $[a, x_1]$.

Theorem 2.5.1 tells us that this solution y^* is a limit of a convergent sequence $T^m y_0^*$:

$$\lim_{m\to\infty} \|T^m y_0^* - y^*\|_{C_{1-\gamma,\log}[a,x_1]} = 0, \tag{4.19}$$

where y_0^* is any function in $C_{1-\gamma,\log}[a, x_1]$. It is convenient to take $y_0^*(x) = y_0(x)$ with $y_0(x)$ defined by (4.17). If we denote by

$$y_m(x) := (T^m y_0^*)(x) = y_0(x) + \left(\mathcal{J}_{a+}^{\alpha} f\left[t, \left(T^{m-1} y_0^*\right)(t)\right]\right)(x), \ m \in \mathbb{N} \tag{4.20}$$

and $y_{m-1}(t) = (T^{m-1} y_0^*)(t)$, then clearly $\lim_{m\to\infty} \|y_m - y^*\|_{C_{1-\gamma,\log}[a,x_1]} = 0$.

Next, we consider the interval $[x_1, b]$. From the equation (4.6) we have

$$\begin{aligned} y(x) &= \tfrac{c}{\Gamma(\gamma)} \left(\log \tfrac{x}{a}\right)^{\gamma-1} + \left(\mathcal{J}_{a+}^{\alpha} f[t, y(t)]\right)(x) \\ &= \tfrac{c}{\Gamma(\gamma)} \left(\log \tfrac{x}{a}\right)^{\gamma-1} + \tfrac{1}{\Gamma(\alpha)} \int_a^x \left(\log \tfrac{x}{t}\right)^{\alpha-1} \tfrac{f[t,y(t)]}{t} dt \\ &= \tfrac{c}{\Gamma(\gamma)} \left(\log \tfrac{x}{a}\right)^{\gamma-1} + \tfrac{1}{\Gamma(\alpha)} \int_a^{x_1} \left(\log \tfrac{x}{t}\right)^{\alpha-1} \tfrac{f[t,y(t)]}{t} dt \\ &\quad + \tfrac{1}{\Gamma(\alpha)} \int_{x_1}^x \left(\log \tfrac{x}{t}\right)^{\alpha-1} \tfrac{f[t,y(t)]}{t} dt. \end{aligned}$$

Since the function $y(t)$ is uniquely defined on the interval $(a, x_1]$, the last integral can be considered as the unknown function, and we rewrite the last equation as

$$y(x) = y_{01}(x) + \frac{1}{\Gamma(\alpha)} \int_{x_1}^{x} \left(\log \frac{x}{t}\right)^{\alpha-1} \frac{f[t, y(t)]}{t} dt, \tag{4.21}$$

where $y_{01}(x)$ is defined by

$$y_{01}(x) = \frac{c}{\Gamma(\gamma)} \left(\log \frac{x}{a}\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_a^{x_1} \left(\log \frac{x}{t}\right)^{\alpha-1} \frac{f[t, y(t)]}{t} dt, \tag{4.22}$$

and is a known function. We note that $y_{01}(x) \in C[x_1, b]$.

Next, we consider the point $x_2 \in (x_1, b]$, $x_2 = x_1 + h_1$, $h_1 > 0$

$$w_2 = \frac{A}{\alpha \Gamma(\alpha)} \left(\log \frac{x_2}{x_1}\right)^{\alpha} < 1. \tag{4.23}$$

$C[x_1, x_2]$ is a complete metric space with the distance given by

$$d(y_1 - y_2) = \|y_1 - y_2\|_{C[x_1,x_2]} = \max_{x\in[x_1,x_2]} |y_1(x) - y_2(x)|. \tag{4.24}$$

The integral equation (4.21) may be written shortly as

$$y(x) = (Ty)(x), \tag{4.25}$$

where the operator (again denoted by T) is given by

$$(Ty)(x) = y_{01}(x) + \frac{1}{\Gamma(\alpha)} \int_{x_1}^{x} \left(\log \frac{x}{t}\right)^{\alpha-1} \frac{f[t, y(t)]}{t} dt. \tag{4.26}$$

As in the first part of this proof, since $y_{01}(x) \in C[x_1, x_2]$ and $f[x, y(x)] \in C_{\mu,\log}[a, b]$ for any $y(x) \in C_{\mu,\log}[a, b]$, then $f[x, y(x)] \in C[x_1, x_2]$ for any $y(x) \in C[x_1, x_2]$, then by Lemma 2.3.4**,** we deduce that the integral in the right-hand side of (4.26) also belongs to $C[x_1, x_2]$, and hence $(Ty)(x) \in C[x_1, x_2]$.

Moreover, using the Lipschitz condition Definition 2.5.1 and applying the Lemma 2.3.4, we find

$$\begin{aligned}
\|Ty_1 - Ty_2\|_{C[x_1,x_2]} &= \left\| \mathcal{J}^{\alpha}_{x_1^+} f[t, y_1(t)] - \mathcal{J}^{\alpha}_{x_1^+} f[t, y_2(t)] \right\|_{C[x_1,x_2]} \\
&\leq \left\| \mathcal{J}^{\alpha}_{x_1^+} |f[t, y_1(t)] - f[t, y_2(t)]| \right\|_{C[x_1,x_2]} \\
&\leq \tfrac{1}{\alpha\Gamma(\alpha)} \left(\log \tfrac{x_2}{x_1}\right)^{\alpha} \|f[t, y_1(t)] - f[t, y_2(t)]\|_{C[x_1,x_2]} \\
&\leq \tfrac{A}{\alpha\Gamma(\alpha)} \left(\log \tfrac{x_2}{x_1}\right)^{\alpha} \|y_1(t) - y_2(t)\|_{C[x_1,x_2]} \\
&\leq w_2 \|y_1(t) - y_2(t)]\|_{C[x_1,x_2]}.
\end{aligned} \tag{4.27}$$

This, together with our assumption $0 < w_2 < 1$, shows that T is a contraction and therefore from Theorem 2.5.1, there exists a unique solution $y_1^*(x) \in C[x_1, x_2]$ to equation (4.6) on the interval $[x_1, x_2]$. Notice that $y_1^*(x_1) = y^*(x_1) = y_{01}(x_1)$. Further, Theorem 2.5.1 guarantees that this solution is the limit of a convergent sequence $T^m y_{01}^*$:

$$\lim_{m\to\infty} \|T^m y_{01}^* - y_1^*\|_{C[x_1,x_2]} = 0 \tag{4.28}$$

where y_{01}^* is any function in $C[x_1, x_2]$, which we can pick $y_{01}^*(x) = y_{01}(x)$ defined by (4.22). Therefore

$$\lim_{m\to\infty} \|y_m - y_1^*\|_{C[x_1,x_2]} = 0, \tag{4.29}$$

where

$$y_m(x) = (T^m y_{01}^*)(x) = y_{01}(x) + \frac{1}{\Gamma(\alpha)} \int_{x_1}^{x} \left(\log \frac{x}{t}\right)^{\alpha-1} \frac{f[t, (T^{m-1} y_{01}^*)(t)]}{t} dt. \tag{4.30}$$

If $x_2 \neq b$, we consider the interval $[x_2, x_3]$, where $x_3 = x_2 + h_2$, $h_2 > 0$ such that $x_3 \leq b$ and

$$w_3 = \frac{A}{\alpha\Gamma(\alpha)} \left(\log \frac{x_3}{x_2}\right)^{\alpha} < 1. \tag{4.31}$$

Using the same arguments as above, we derive that there exists a unique solution $y_2^* \in C[x_2, x_3]$ to the equation (4.6) on the interval $[x_2, x_3]$. If $x_3 \neq b$, then we continue the process until we reach a solution y to (4.6), $y(x) = y_k^*(x)$, and $y_k^* \in C[x_k, x_{k+1}]$ $(k = 1, ..., L)$, where $a = x_0 < x_1 < ... < x_{L+1}$ and

$$w_{k+1} = \frac{A}{\alpha\Gamma(\alpha)} \left(\log \frac{x_{k+1}}{x_k}\right)^{\alpha} < 1, \tag{4.32}$$

Assume that $b-a>\frac{\Gamma(\alpha+\gamma)}{A\Gamma(\gamma)}$ (for otherwise take $x_1=b$). Then, divide the length of the interval $\left[a+\frac{\Gamma(\alpha+\gamma)}{A\Gamma(\gamma)},b\right]$ by $\left(\frac{\alpha\Gamma(\alpha)}{A}\right)^{1/\alpha}$. Let M be that quotient. It appears that for $L=[M]+1$ and b is reached after a finite number of steps, $x_{L+1}=b$. Then, there exists a unique solution $y\left(x\right)\in C\left[x_1,b\right]$ to equation (4.6) on the interval $[x_1,b]$.

Putting together the solutions in $[a,x_1]$ and $[x_1,b]$ and taking into account the Lemma 2.3.5, we obtain that there exists a unique solution $y\in C_{1-\gamma,\log}\left[a,b\right]$ to the Volterra integral equation (4.6) on the whole interval $[a,b]$. Hence $y\in C_{1-\gamma,\log}\left[a,b\right]$ is the unique solution to the Cauchy-type problem (4.2).

It remains to show that such a unique solution $y\in C_{1-\gamma,\log}\left[a,b\right]$ is actually in $C^{\alpha,\beta}_{\delta;1-\gamma,\mu}\left[a,b\right]$. To this end we need to prove that $\mathcal{D}^{\alpha,\beta}_{a^+}y\in C_{\mu,\log}\left[a,b\right]$. Let us recall that our y is a limit of the sequence y_m, where $y_m\left(x\right)=\left(T^m y_0^*\right)\in C_{1-\gamma,\log}\left[a,b\right]$, that is

$$\lim_{m\to\infty}\left\|y_m-y\right\|_{C_{1-\gamma,\log}[a,b]}=0, \tag{4.33}$$

with a certain choice of $y_0^*\left(x\right)$ on each $\left[a,x_1\right],...,\left[x_L,b\right]$.

If $y_0\left(x\right)\neq 0$, then we can take $y_0^*\left(x\right)=y_0\left(x\right)$. Since $\mu\geq 1-\gamma$, then by (4.2), the Lipschitz condition Definition 2.5.1 and Property 2.1.1, we have

$$\begin{aligned}\left\|\mathcal{D}^{\alpha,\beta}_{a^+}y_m-\mathcal{D}^{\alpha,\beta}_{a^+}y\right\|_{C_{\mu,\log}[a,b]} &= \left\|f[x,y_m]-f[x,y]\right\|_{C_{\mu,\log}[a,b]}\\ \leq A\left\|y_m-y\right\|_{C_{\mu,\log}[a,b]} &\leq A\left(\log\tfrac{b}{a}\right)^{\mu-1+\gamma}\left\|y_m-y\right\|_{C_{1-\gamma,\log}[a,b]}.\end{aligned} \tag{4.34}$$

In virtue of (4.33) and (4.34) it follows that

$$\lim_{m\to\infty}\left\|\mathcal{D}^{\alpha,\beta}_{a^+}y_m-\mathcal{D}^{\alpha,\beta}_{a^+}y\right\|_{C_{\mu,\log}[a,b]}=0. \tag{4.35}$$

We entail from this relation that $\left(\mathcal{D}_{a^+}^{\alpha,\beta}y\right)(x) \in C_{\mu,\log}[a,b]$ if $\left(\mathcal{D}_{a^+}^{\alpha,\beta}y_m\right)(x) \in C_{\mu,\log}[a,b]$, $m = 1, 2, \ldots$. This latter property holds in as much as $\mathcal{D}_{a^+}^{\alpha,\beta}y_m(x) = f[x, y_{m-1}(x)]$ and $f[x, y(x)] \in C_{\mu,\log}[a,b]$ for any $y(x) \in C_{\mu,\log}[a,b]$. Hence a unique solution $y(x) \in C_{1-\gamma,\log}[a,b]$ has the property $\left(\mathcal{D}_{a^+}^{\alpha,\beta}y\right)(x) \in C_{\mu,\log}[a,b]$. Consequently, $y(x) \in C_{\delta;1-\gamma,\mu}^{\alpha,\beta}[a,b]$. This completes the proof of Theorem 4.2.1.

4.3 Stability of Hilfer-Hadamard Fractional Derivative

In this section we consider the weighted Cauchy-type problem

$$\begin{cases} \mathcal{D}_{a^+}^{\alpha,\beta}u(t) = f(t,u), \quad t > a > 0, \\ \left(\log \frac{t}{a}\right)^{(1-\beta)(1-\alpha)} u(t)\,|_{t=a} = b, \end{cases} \tag{4.36}$$

where $\mathcal{D}_{+}^{\alpha,\beta}$ is the Hilfer-Hadamard fractional derivative (**HHFD**) of order $0 < \alpha < 1$ and type $0 \leq \beta \leq 1$ and $b \in \mathbb{R}^*$ (the set of all real numbers except 0). We will assume the following hypotheses on the function $f(t,u)$:

(F*) $f(t,u)$ is a continuous nonlinear function on $(a,\infty) \times \mathbb{R}$ and is such that

$$|f(t,u)| \leq \left(\log \frac{t}{a}\right)^{\mu} \varphi(t)\,|u(t)|^m, \quad \mu \geq 0, \quad m > 1, \quad t \geq a$$

where φ is a continuous function on $[a,\infty)$.

For this we need the following inequality.

Lemma 4.3.1 : *If* $\lambda,\ \nu,\ \omega > 0$, *then for any* $t > a$, $a > 0$ *we have*

$$\left(\log\frac{t}{a}\right)^{1-\nu}\int_a^t\left(\log\frac{t}{s}\right)^{\nu-1}\left(\log\frac{s}{a}\right)^{\lambda-1}\left(\frac{s}{a}\right)^{-\omega}\frac{ds}{s}\leq C\omega^{-\lambda},$$

where C *is a positive constant independent of* t.

Proof : Let us denote by $I(t)$ the left-hand side of the inequality in the Lemma. We consider the change of variable, $\xi = \frac{\log(s/a)}{\log(t/a)}$ then $\frac{s}{a} = \left(\frac{t}{a}\right)^{\xi}$ and $\log\frac{t}{s} = (1-\xi)\left(\log\frac{t}{a}\right)$. It follows that

$$I(t) = \left(\log\frac{t}{a}\right)^{\lambda}\int_0^1(1-\xi)^{\nu-1}\xi^{\lambda-1}\left(\frac{t}{a}\right)^{-\omega\xi}d\xi$$

or

$$I(t) = \left(\log\frac{t}{a}\right)^{\lambda}\int_0^1(1-\xi)^{\nu-1}\xi^{\lambda-1}\exp\left(-\omega\xi\log\left(\frac{t}{a}\right)\right)d\xi.$$

Observe that, for $\xi\geq 1$ and $[\lambda]+1\geq\lambda$ we have $\xi^{[\lambda]+1}\geq\xi^{\lambda}$. Also since $\lambda+2\geq[\lambda]+2$ and the Gamma function is increasing for $[2,\infty)$, we have $\Gamma(\lambda+2)\geq\Gamma([\lambda]+2)$ or $\frac{1}{\Gamma([\lambda]+2)}\geq\frac{1}{\Gamma(\lambda+2)}$. Moreover $e^{\xi}\geq\frac{\xi^{[\lambda]+1}}{\Gamma([\lambda]+2)}$ $(\Gamma([\lambda]+2) = ([\lambda]+1)!)$, and hence

$$e^{\xi}\geq\frac{\xi^{[\lambda]+1}}{\Gamma([\lambda]+2)}\geq\frac{\xi^{\lambda}}{\Gamma([\lambda]+2)}\geq\frac{\xi^{\lambda}}{\Gamma(\lambda+2)}$$

or

$$e^{-\xi}\leq\frac{\Gamma(\lambda+2)}{\xi^{\lambda}}.$$

Therefore, for $0 \le \xi < 1/2$ we get

$$(1-\xi)^{\nu-1} \le \max\left(1, 2^{1-\nu}\right).$$

For $1/2 < \xi \le 1$ we have

$$\exp\left(-\omega\xi \log\left(\frac{t}{a}\right)\right) \le \frac{\Gamma(\lambda+2)}{\left(\omega\xi\log\left(\frac{t}{a}\right)\right)^{\lambda}} \le \frac{\omega^{-\lambda}}{\xi}\Gamma(\lambda+2) \le 2\omega^{-\lambda}\Gamma(\lambda+2).$$

This means that

$$\left(\log\frac{t}{a}\right)^{\lambda}(1-\xi)^{\nu-1}\xi^{\lambda-1}\exp\left(-\omega\xi\log\left(\frac{t}{a}\right)\right)$$

$$\le \begin{cases} \max\left(1, 2^{1-\nu}\right)\left(\log\frac{t}{a}\right)^{\lambda}\xi^{\lambda-1}\exp\left(-\omega\xi\log\left(\frac{t}{a}\right)\right) & for \quad 0 \le \xi < 1/2, \\ 2(1-\xi)^{\nu-1}\Gamma(\lambda+2)\,\omega^{-\lambda} & for \quad 1/2 < \xi \le 1. \end{cases}$$

Consequently

$$I(t) \le \max\left(1, 2^{1-\nu}\right)\left(\log\frac{t}{a}\right)^{\lambda}\int_0^{1/2}\xi^{\lambda-1}\exp\left(-\omega\xi\log\left(\frac{t}{a}\right)\right)d\xi$$

$$+2\omega^{-\lambda}\Gamma(\lambda+2)\int_{1/2}^{1}(1-\xi)^{\nu-1}\,d\xi.$$

Let $u = \omega\xi\log\left(\frac{t}{a}\right)$, we see that

$$I(t) \le \max\left(1, 2^{1-\nu}\right)\left(\log\frac{t}{a}\right)^{\lambda}\int_0^{\infty}\left(\frac{u}{\omega\log\left(\frac{t}{a}\right)}\right)^{\lambda-1}e^{-u}\frac{du}{\omega\log\left(\frac{t}{a}\right)}$$

$$+2\omega^{-\lambda}\Gamma(\lambda+2)\left[-\frac{(1-\xi)^{\nu}}{\nu}\right]_{\xi=1/2}^{1}.$$

Thus

$$I(t)\leq \max\left(1,2^{1-\nu}\right)\omega^{-\lambda}\Gamma(\lambda)+\frac{2^{1-\nu}\omega^{-\lambda}\Gamma(\lambda+2)}{\nu}.$$

As a result, $I(t)\leq \max\{1,2^{1-\nu}\}\Gamma(\lambda)(1+\lambda(\lambda+1)/\nu)\omega^{-\lambda}$.

For $0<\xi<1$, $e^{\xi}\geq 1$ it is clear that

$$\Gamma(\lambda+2)e^{\xi}\geq 1\geq \xi^{\lambda}$$

holds and we proceed in the same manner to conclude that

$$\left(\log\frac{t}{a}\right)^{1-\nu}\int_a^t\left(\log\frac{t}{s}\right)^{\nu-1}\left(\log\frac{s}{a}\right)^{\lambda-1}\left(\frac{s}{a}\right)^{-\omega}\frac{ds}{s}\leq C\omega^{-\lambda},$$

where $C=\max\{1,2^{1-\nu}\}\Gamma(\lambda)(1+\lambda(\lambda+1)/\nu)$. The proof is complete.

Let p and q be conjugate exponents, i.e. $pq=p+q$, and let $\lambda_1:=1+p[\mu-(1-\gamma)m]$ and $\lambda_2:=1+p(\alpha-1)$, where $\gamma:=\alpha+\beta-\alpha\beta$. If $\mu-(m-1)(1-\gamma)>0$ and $q>1/\alpha$, then $\lambda_1>0$ and $\lambda_2>0$. We denote by $\mathbf{L}^*$ the positive real number

$$\mathbf{L}^*:=\left(\frac{\Gamma(\alpha)}{2^{m+(\alpha-1)}|b|^{m-1}}\right)^m\left(\frac{(2a)^m}{m-1}\right)^{1/q}\left[\frac{(p-1)^{\lambda_1}}{\Gamma(\lambda_1)(1+\lambda_1/\lambda_2)}\right]^{m/p},\ m>1.$$

Theorem 4.3.1: *Suppose that* $f(t,u)$ *satisfies* ***(F*)*** *and* $\mu > (m-1)(1-\gamma)$. *If*

$$\left(\|\varphi(t)\|_q\right)^{m-1}\left\|\varphi(t)\left(\log\frac{t}{a}\right)^{-m\beta(1-\alpha)}\right\|_q < \mathbf{L}^*$$

for some $q > 1/\alpha$, *then there exists a positive constant* C *such that* $|u(t)| \leq C\left(\log\frac{t}{a}\right)^{\gamma-1}$, $t > ae$ $(a > 0)$, *where* $\gamma = \alpha+\beta-\alpha\beta$.

Proof : Let us consider the Volterra integral equation

$$u(t) = b\left(\log\frac{t}{a}\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_a^t\left(\log\frac{t}{s}\right)^{\alpha-1} f\left[s,u(s)\right]\frac{ds}{s}, \quad t > a > 0 \tag{4.37}$$

associated to problem (4.36). Multiplying both sides of (4.37) by $\left(\log\frac{t}{a}\right)^{1-\gamma}$ and using the assumption *(F*)* on $f(t,u)$, we get

$$\left(\log\frac{t}{a}\right)^{1-\gamma}|u(t)| \leq |b| + \frac{\left(\log\frac{t}{a}\right)^{1-\gamma}}{\Gamma(\alpha)}\int_a^t\left(\log\frac{t}{s}\right)^{\alpha-1}\left(\log\frac{s}{a}\right)^{\mu}\varphi(s)|u(s)|^m\frac{ds}{s}. \tag{4.38}$$

Let $v(t)$ denote the left-hand side of (4.38). The insertion of the term

$$\left(\log\frac{s}{a}\right)^{(1-\gamma)m}\left(\log\frac{s}{a}\right)^{-(1-\gamma)m}$$

inside the integral gives

$$v(t) \leq |b| + \frac{\left(\log\frac{t}{a}\right)^{1-\gamma}}{\Gamma(\alpha)}\int_a^t\left(\log\frac{t}{s}\right)^{\alpha-1}\left(\log\frac{s}{a}\right)^{\mu-(1-\gamma)m}\varphi(s)v^m(s)\frac{ds}{s}, \quad t > a. \tag{4.39}$$

Now the Hölder inequality with exponents p and q yields

$$\int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} \left(\log \frac{s}{a}\right)^{\mu-(1-\gamma)m} \varphi(s)\, v^m(s) \frac{ds}{s}$$

$$\leq \left(\int_a^t \left(\log \frac{t}{s}\right)^{p(\alpha-1)} \left(\log \frac{s}{a}\right)^{p(\mu-(1-\gamma)m)} \frac{ds}{s^p}\right)^{1/p} \left(\int_a^t \varphi^q(s)\, v^{qm}(s)\, ds\right)^{1/q},$$

or

$$\int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} \left(\log \frac{s}{a}\right)^{\mu-(1-\gamma)m} \varphi(s)\, v^m(s) \frac{ds}{s}$$

$$\leq a^{-1/q} \left(\int_a^t \left(\log \frac{t}{s}\right)^{p(\alpha-1)} \left(\log \frac{s}{a}\right)^{p(\mu-(1-\gamma)m)} \left(\frac{s}{a}\right)^{-(p-1)} \frac{ds}{s}\right)^{1/p}$$

$$\times \left(\int_a^t \varphi^q(s)\, v^{qm}(s)\, ds\right)^{1/q}, \quad t > a.$$

Since $\lambda_1 - 1 = p[\mu - (1-\gamma)m]$, $\lambda_2 - 1 = p(\alpha - 1)$ and $\lambda_1,\ \lambda_2,\ p-1 > 0$, we may apply Lemma 4.3.1 (with ν replaced by λ_2, λ replaced by λ_1 and ω replaced by $p-1$) to get

$$\int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} \left(\log \frac{s}{a}\right)^{\mu-(1-\gamma)m} \varphi(s)\, v^m(s) \frac{ds}{s} \leq a^{-1/q} C_1 \left(\log \frac{t}{a}\right)^{\alpha-1} \tag{4.40}$$

$$\times \left(\int_a^t \varphi^q(s)\, v^{qm}(s)\, ds\right)^{\frac{1}{q}}, \quad t > a$$

where C_1 is the constant appearing in Lemma 4.3.1 corresponding to the present exponents. That is

$$C_1 = \left(2^{p(\alpha-1)} \Gamma(\lambda_1)\left(1 + \lambda_1(\lambda_1+1)/\lambda_2\right)(p-1)^{-\lambda_1}\right)^{1/p}.$$

Combining (4.39) and (4.40) we entail that

$$v(t) \leq |b| + \hat{C}_1 \left(\log \frac{t}{a}\right)^{-\beta(1-\alpha)} \left(\int_a^t \varphi^q(s)\, v^{qm}(s)\, ds\right)^{1/q}, \quad t > a \tag{4.41}$$

where $\hat{C}_1 = a^{-1/q} \frac{C_1}{\Gamma(\alpha)}$. Multiplying both sides of (4.41) by $\left(\log \frac{t}{a}\right)^{\beta(1-\alpha)}$, we obtain

$$\left(\log \frac{t}{a}\right)^{\beta(1-\alpha)} v(t) \leq |b| \left(\log \frac{t}{a}\right)^{\beta(1-\alpha)} + \hat{C}_1 \left(\int_a^t \varphi^q(s)\, v^{qm}(s)\, ds\right)^{1/q}, \quad t > a. \tag{4.42}$$

Let $z(t)$ denote the left-hand side of (4.42). The insertion of the term

$$\left(\log \frac{s}{a}\right)^{-qm\beta(1-\alpha)} \left(\log \frac{s}{a}\right)^{qm\beta(1-\alpha)}$$

inside the integral gives

$$z(t) \leq |b| \left(\log \frac{t}{a}\right)^{\beta(1-\alpha)} + \hat{C}_1 \left(\int_a^t \varphi^q(s) \left(\log \frac{s}{a}\right)^{-qm\beta(1-\alpha)} z^{qm}(s)\, ds\right)^{1/q}. \tag{4.43}$$

Raising both sides of (4.43) to the power q and using Lemma 2.5.2, we get

$$z^q(t) \leq 2^{q-1} \left(|b|^q \left(\log \frac{t}{a}\right)^{q\beta(1-\alpha)} + \hat{C}_1^q \int_a^t \varphi^q(s) \left(\log \frac{s}{a}\right)^{-qm\beta(1-\alpha)} z^{qm}(s)\, ds\right). \tag{4.44}$$

Let us set

$$w(t) = \hat{C}_1^q \int_a^t \varphi^q(s) \left(\log \frac{s}{a}\right)^{-qm\beta(1-\alpha)} z^{qm}(s)\, ds, \quad t > a. \tag{4.45}$$

Then, clearly $w(a) = 0$, and by differentiation

$$w'(t) = \hat{C}_1^q \varphi^q(t) \left(\log \frac{t}{a}\right)^{-qm\beta(1-\alpha)} z^{qm}(t), \quad t > a. \tag{4.46}$$

Moreover, it is clear that as φ and v are nonnegative continuous functions in $[a, \infty)$. Thus w is a continuous, nonnegative and nondecreasing function in $[a, \infty)$.

Now, we would like to estimate the right hand side of (4.46) in term of $w(t)$. From (4.44) and (4.45) we entail that

$$z^q(t) \leq 2^{q-1} \left(|b|^q \left(\log \frac{t}{a}\right)^{q\beta(1-\alpha)} + w(t) \right), \quad t > a. \tag{4.47}$$

Raising both sides of (4.47) to the power m and using Lemma 2.5.2 again, we get

$$z^{qm}(t) \leq 2^{mq-1} \left(|b|^{mq} \left(\log \frac{t}{a}\right)^{mq\beta(1-\alpha)} + w^m(t) \right), \quad t > a. \tag{4.48}$$

The substitution of (4.48) in (4.46) yields

$$w'(t) \leq 2^{mq-1} \hat{C}_1^q \varphi^q(t) \left(\log \frac{t}{a}\right)^{-qm\beta(1-\alpha)} \left(|b|^{mq} \left(\log \frac{t}{a}\right)^{mq\beta(1-\alpha)} + w^m(t) \right)$$

$$\leq 2^{mq-1} |b|^{mq} \hat{C}_1^q \varphi^q(t) + 2^{mq-1} \hat{C}_1^q \left(\log \frac{t}{a}\right)^{-qm\beta(1-\alpha)} \varphi^q(t) w^m(t). \tag{4.49}$$

Applying Lemma 2.5.3 (with $w(u) = u^m$) we infer that

$$w(t) \leq G^{-1} \left[G \left(w(a) + \int_a^t 2^{mq-1} |b|^{mq} \hat{C}_1^q \varphi^q(s)\, d \right) s \right.$$

$$+\int_a^t 2^{mq-1}\hat{C}_1^q \left(\log\frac{s}{a}\right)^{-qm\beta(1-\alpha)} \varphi^q(s)\,ds\Bigg].$$

Let us set

$$l(t) = 2^{mq-1}|b|^{mq}\,\hat{C}_1^q \int_a^t \varphi^q(s)\,ds$$

and

$$k(t) = 2^{mq-1}\hat{C}_1^q \int_a^t \left(\log\frac{s}{a}\right)^{-qm\beta(1-\alpha)} \varphi^q(s)\,ds,$$

then

$$w(t) \le G^{-1}\left[G(l(t)) + k(t)\right],$$

where we have used the fact that $w(a) = 0$. Since $G(r) = \int_{r_0}^r \frac{ds}{s^m}$, $r > 0$, $r_0 > 0$, then $G(r) = \frac{r^{1-m}}{1-m} - \frac{r_0^{1-m}}{1-m}$ and $G^{-1}(y) = \left[r_0^{1-m} - (m-1)y\right]^{-\frac{1}{m-1}}$. That is

$$w(t) \le G^{-1}\left[\frac{l(t)^{1-m}}{1-m} - \frac{l(t_0)^{1-m}}{1-m} + k(t)\right]$$

$$\le \left[l(t_0)^{1-m} - (m-1)\left(\frac{l(t)^{1-m}}{1-m} - \frac{l(t_0)^{1-m}}{1-m} + k(t)\right)\right]^{-\frac{1}{m-1}}$$

$$\le \left[l(t)^{1-m} - (m-1)\,k(t)\right]^{-\frac{1}{m-1}}$$

as long as

$$l(t)^{m-1}k(t) < \frac{1}{m-1}.$$

In particular, if $\left(\|\varphi(t)\|_q\right)^{m-1}\left\|\varphi(t)\left(\log\frac{t}{a}\right)^{-m\beta(1-\alpha)}\right\|_q < \mathbf{L}^*/\mathbf{2}$, then $w(t) \le K_1$ for some positive constant K_1 for all $t > a$, and thus from (4.43) we see that

$$z(t) \leq |b| \left(\log \frac{t}{a} \right)^{\beta(1-\alpha)} + K_1^{1/q},$$

and then

$$v(t) \leq |b| + K_1^{1/q} \left(\log \frac{t}{a} \right)^{-\beta(1-\alpha)} \leq C, \quad t \geq t_0 > ae$$

for some positive constant C. This yields that $|u(t)| \leq C \left(\log \frac{t}{a} \right)^{\gamma-1}$ for $t \geq t_0 > ae$. The proof is complete.

4.4 Non-existence result

We consider the Cauchy-type problem

$$\begin{cases} \left(\mathcal{D}_{a^+}^{\alpha,\beta} u \right)(t) = f[t, u(t)], & t > a \\ \left(\mathcal{D}_{a^+}^{(\beta-1)(1-\alpha)} u \right)(a) = b \end{cases} \tag{4.50}$$

where

$$\left(\mathcal{D}_{a^+}^{\alpha,\beta} u \right)(t) = \mathcal{J}_{a^+}^{\beta(1-\alpha)} \left(t \frac{d}{dt} \right) \left(\mathcal{J}_{a^+}^{(1-\beta)(1-\alpha)} u \right)(t)$$

is the Hilfer-Hadamard Fractional Derivative (**HHFD**) of order $0 < \alpha < 1$ and type $0 \leq \beta \leq 1$.

We investigate the case when $f[t, u(t)] \geq \left(\log \frac{t}{a} \right)^{\mu} |u(t)|^m$ for some $m > 1$ and $\mu \in \mathbb{R}$.

That is we consider the Cauchy problem

$$\begin{cases} \left(\mathcal{D}_{a^+}^{\alpha,\beta}u\right)(t) \geq \left(\log \frac{t}{a}\right)^{\mu} |u(t)|^m, \quad t > a > 0,\ m > 1,\ \mu \in \mathbb{R} \\ \left(\mathcal{D}_{a^+}^{\gamma-1}u\right)(a) = b \geq 0 \end{cases} \tag{4.51}$$

where $\mathcal{D}_{a^+}^{\alpha,\beta}$ is the **HHFD** of order $0 < \alpha < 1$ and type $0 \leq \beta \leq 1$ and $\gamma = \alpha + \beta - \alpha\beta$.

For this we need the following lemma.

Lemma 4.4.1: *If* $\alpha > 0$ *and* $f \in C[a,b]$, *then*

$$\left(\mathcal{J}_{a+}^{\alpha}f\right)(a) = \lim_{t\to a}\left(\mathcal{J}_{a+}^{\alpha}f\right)(t) = 0$$

and

$$\left(\mathcal{J}_{b^-}^{\alpha}f\right)(b) = \lim_{t\to b}\left(\mathcal{J}_{b^-}^{\alpha}f\right)(t) = 0.$$

Proof : Since $f \in C[a,b]$, then on $[a,b]$, we have

$$|f(t)| < M,$$

for some positive constant M.

Therefore

$$\begin{aligned} \left|\left(\mathcal{J}_{a+}^{\alpha}f\right)(t)\right| &\leq \frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1} |f(s)| \frac{ds}{s} \\ &\leq \frac{M}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1} \frac{ds}{s} \\ &\leq \frac{M}{\alpha\Gamma(\alpha)}\left[-\left(\log\frac{t}{s}\right)^{\alpha}\right]_{s=a}^{t} = \frac{M}{\Gamma(\alpha+1)}\left(\log\frac{t}{a}\right)^{\alpha}. \end{aligned}$$

As $\alpha > 0$ we see that

$$\left(\mathcal{J}_{a+}^{\alpha} f\right)(a) = \lim_{t \to a} \left(\mathcal{J}_{a+}^{\alpha} f\right)(t) = 0.$$

Similarity

$$\begin{aligned}
\left|\left(\mathcal{J}_{b^-}^{\alpha} f\right)(t)\right| &\leq \frac{1}{\Gamma(\alpha)} \int_t^b \left(\log \frac{s}{t}\right)^{\alpha-1} |f(s)| \frac{ds}{s} \\
&\leq \frac{M}{\Gamma(\alpha)} \int_t^b \left(\log \frac{s}{t}\right)^{\alpha-1} \frac{ds}{s} \\
&\leq \frac{M}{\alpha\Gamma(\alpha)} \left[\left(\log \frac{s}{t}\right)^{\alpha}\right]_{s=t}^{b} = \frac{M}{\Gamma(\alpha+1)} \left(\log \frac{b}{t}\right)^{\alpha}.
\end{aligned}$$

As $\alpha > 0$ we see that

$$\left(\mathcal{J}_{b^-}^{\alpha} f\right)(b) = \lim_{t \to b} \left(\mathcal{J}_{b^-}^{\alpha} f\right)(t) = 0.$$

Theorem 4.4.1: *Assume that $\mu \geq 0$. Then, Problem (4.51) does not admit global nontrivial solution in $C_{1-\gamma,\log}^{\gamma}[a,b]$ when $b \geq 0$.*

Proof : Assume that a nontrivial solution exists for all time $t > a$. Let $\varphi(t) \in C^1([a,\infty))$ be a test function satisfying : $\varphi(t) \geq 0$, $\varphi(t)$ is non-increasing and such that

$$\varphi(t) := \begin{cases} 1, & a \leq t \leq \theta T \\ 0, & t \geq T \end{cases}$$

for some $T > a$ and some θ $(\theta < 1)$ such that $a < \theta T < T$. Multiplying the inequality in (4.51) by $\frac{\varphi(t)}{t}$ and integrating over $[a,T]$ we get

$$\int_a^T \varphi(t) \left(\mathcal{D}_{a^+}^{\alpha,\beta} u\right)(t) \frac{dt}{t} \geq \int_a^T \left(\log \frac{t}{a}\right)^{\mu} |u(t)|^m \varphi(t) \frac{dt}{t}$$

and from the definition of $\left(\mathcal{D}_{a^+}^{\alpha,\beta}u\right)(t)$ we can write

$$\int_a^T \varphi(t)\left(\mathcal{J}_{a^+}^{\beta(1-\alpha)} t\frac{d}{dt}\mathcal{J}_{a^+}^{1-\gamma}u\right)(t)\frac{dt}{t} \geq \int_a^T \left(\log\frac{t}{a}\right)^{\mu} |u(t)|^m \varphi(t)\frac{dt}{t}. \tag{4.52}$$

By virtue of Lemma 2.3.6 (after extending by zero outside $[a,T]$), we may deduce from (4.52) that

$$\int_a^T \frac{d}{dt}\left(\mathcal{J}_{a^+}^{1-\gamma}u\right)(t)\left(\mathcal{J}_{T^-}^{\beta(1-\alpha)}\varphi(t)\right)(t)\,dt \geq \int_a^T \left(\log\frac{t}{a}\right)^{\mu} |u(t)|^m \varphi(t)\frac{dt}{t}. \tag{4.53}$$

Notice that Lemma 2.3.6 is valid in our case since $(\left(\log\frac{t}{a}\right)^{(1-\gamma)}\left(\mathcal{D}_{a^+}^{\gamma}u\right) \in C[a,T]$ implies that $\left|\left(\log\frac{t}{a}\right)^{(1-\gamma)}\left(\mathcal{D}_{a^+}^{\gamma}u\right)(t)\right| \leq M$ on $[a,T]$ for some positive constant $M)$

$$\int_a^T \left|t^{-\frac{1}{p}}\left(\mathcal{D}_{a^+}^{\gamma}u\right)(t)\right|^{p'}\frac{dt}{t} \leq M\int_a^T t^{1-p'}\left(\log\frac{t}{a}\right)^{-p'(1-\gamma)}\frac{dt}{t}$$

$$\leq M\int_a^{\infty} t^{1-p'}\left(\log\frac{t}{a}\right)^{-p'(1-\gamma)}\frac{dt}{t}.$$

Let $s = (p'-1)\left(\log\frac{t}{a}\right)$, then by the definition of the Gamma function

$$\int_a^T \left|t^{-\frac{1}{p}}\left(\mathcal{D}_{a^+}^{\gamma}u\right)(t)\right|^{p'}\frac{dt}{t} \leq \frac{Ma^{1-p'}}{(p'-1)^{1-p'(1-\gamma)}}\int_0^{\infty} s^{-p'(1-\gamma)}e^{-s}ds$$

$$\leq \frac{Ma^{1-p'}}{(p'-1)^{1-p'(1-\gamma)}}\Gamma\left(1-p'(1-\gamma)\right) < \infty.$$

Hence $t\frac{d}{dt}\left(\mathcal{J}_{a^+}^{1-\gamma}u\right)t = \left(\mathcal{D}_{a^+}^{\gamma}u\right)(t) \in X_{-1/p}^{p'}$ (and $\varphi \in L_p$) for some $p > \frac{1}{\gamma}$.

An integration by parts in (4.53) yields

$$\left[\left(\mathcal{J}_{a^+}^{1-\gamma}u\right)(t)\left(\mathcal{J}_{T^-}^{\beta(1-\alpha)}\varphi\right)(t)\right]_{t=a}^{T} - \int_a^T \left(\mathcal{J}_{a^+}^{1-\gamma}u\right)(t)\frac{d}{dt}\left(\mathcal{J}_{T^-}^{\beta(1-\alpha)}\varphi\right)(t)\,dt$$

$$\geq \int_a^T \left(\log\frac{t}{a}\right)^{\mu}|u(t)|^m\varphi(t)\frac{dt}{t}$$

or

$$-b\left(\mathcal{J}_{T^-}^{\beta(1-\alpha)}\varphi\right)(a^+) - \int_a^T \left(\mathcal{J}_{a^+}^{1-\gamma}u\right)(t)\frac{d}{dt}\left(\mathcal{J}_{T^-}^{\beta(1-\alpha)}\varphi\right)(t)\,dt \tag{4.54}$$

$$\geq \int_a^T \left(\log\frac{t}{a}\right)^{\mu}|u(t)|^m\varphi(t)\frac{dt}{t}$$

because $\left(\mathcal{J}_{T^-}^{\beta(1-\alpha)}\varphi\right)(T) = 0$ (see Lemma 4.4.1) and

$$\left(\mathcal{J}_{a^+}^{1-\gamma}u\right)(a) = \left(\mathcal{D}_{a^+}^{\gamma-1}u\right)(a) = b.$$

Multiplying by $\frac{t}{t}$ inside the integral in the left hand side of (4.54) we see that

$$-b\left(\mathcal{J}_{T^-}^{\beta(1-\alpha)}\varphi\right)(a) + \int_a^T \left(\mathcal{J}_{a^+}^{1-\gamma}u\right)(t)\left(-t\frac{d}{dt}\right)\left(\mathcal{J}_{T^-}^{\beta(1-\alpha)}\varphi\right)(t)\frac{dt}{t}$$

$$\geq \int_a^T \left(\log\frac{t}{a}\right)^{\mu}|u(t)|^m\varphi(t)\frac{dt}{t}.$$

It appears from Definition 2.3.4 that

$$-b\left(\mathcal{J}_{T^-}^{\beta(1-\alpha)}\varphi\right)(a^+) + \int_a^T \left(\mathcal{J}_{a^+}^{1-\gamma}u\right)(t)\left(\mathcal{D}_{T^-}^{1-\beta(1-\alpha)}\varphi\right)(t)\frac{dt}{t}$$

$$\geq \int_a^T \left(\log\frac{t}{a}\right)^{\mu} |u(t)|^m \varphi(t)\frac{dt}{t}$$

and from Lemma 2.3.3 we see that

$$-b\left(\mathcal{J}_{T^-}^{\beta(1-\alpha)}\varphi\right)(a^+) + \int_a^T \left(\mathcal{J}_{a^+}^{1-\gamma}u\right)(t)\left[\frac{\varphi(T)}{\Gamma(\beta(1-\alpha))}\left(\log\frac{T}{t}\right)^{\beta(1-\alpha)-1}\right.$$

$$\left.-\frac{1}{\Gamma(\beta(1-\alpha))}\int_t^T\left(\log\frac{s}{t}\right)^{\beta(1-\alpha)-1}\varphi'(s)\,ds\right]\frac{dt}{t} \geq \int_a^T \left(\log\frac{t}{a}\right)^{\mu}|u(t)|^m\varphi(t)\frac{dt}{t}.$$

Since $\varphi(T) = 0$ and

$$\frac{1}{\Gamma(\beta(1-\alpha))}\int_t^T\left(\log\frac{s}{t}\right)^{\beta(1-\alpha)-1}\varphi'(s)\,ds = \left(\mathcal{J}_{T^-}^{\beta(1-\alpha)}\delta\varphi\right)(t),$$

the last inequality becomes

$$-b\left(\mathcal{J}_{T^-}^{\beta(1-\alpha)}\varphi\right)(a^+) - \int_a^T \left(\mathcal{J}_{a^+}^{1-\gamma}u\right)(t)\left(\mathcal{J}_{T^-}^{\beta(1-\alpha)}\delta\varphi\right)(t)\frac{dt}{t}$$

$$\geq \int_a^T \left(\log\frac{t}{a}\right)^{\mu}|u(t)|^m\varphi(t)\frac{dt}{t}.$$

Note that $\delta\varphi \in L_p$ and by the same argument as the one used at the beginning of the proof we may show that $\mathcal{J}_{a^+}^{1-\gamma}u \in X_{-1/p}^{p'}$ since $\mathcal{J}_{a^+}^{1-\gamma}u \in C_{1-\gamma,\log}[a,T]$. Therefore, Lemma 2.3.6 again allows us to write

$$-b\left(\mathcal{J}_{T^-}^{\beta(1-\alpha)}\varphi\right)(a^+) - \int_a^T \delta\varphi(t)\left(\mathcal{J}_{a^+}^{\beta(1-\alpha)}\mathcal{J}_{a^+}^{1-\gamma}u\right)(t)\frac{dt}{t}$$

$$\geq \int_a^T \left(\log\frac{t}{a}\right)^{\mu}|u(t)|^m\varphi(t)\frac{dt}{t},$$

and by the Semigroup Property 3.3.2

$$-b\left(\mathcal{J}_{T^{-}}^{\beta(1-\alpha)}\varphi\right)\left(a^{+}\right)-\int_{a}^{T}\delta\varphi\left(t\right)\left(\mathcal{J}_{a^{+}}^{1-\alpha}u\right)\left(t\right)\frac{dt}{t}\geq\int_{a}^{T}\left(\log\frac{t}{a}\right)^{\mu}\left|u\left(t\right)\right|^{m}\varphi\left(t\right)\frac{dt}{t}. \tag{4.55}$$

On the other hand

$$\int_{a}^{T}\delta\varphi\left(t\right)\left(\mathcal{J}_{a^{+}}^{1-\alpha}u\right)\left(t\right)\frac{dt}{t}=\frac{1}{\Gamma\left(1-\alpha\right)}\int_{a}^{T}\delta\varphi\left(t\right)\int_{a}^{t}\left(\log\frac{t}{s}\right)^{-\alpha}\frac{u\left(s\right)}{s}ds\frac{dt}{t}$$

$$\leq\frac{1}{\Gamma\left(1-\alpha\right)}\int_{a}^{T}\left|\delta\varphi\left(t\right)\right|\int_{a}^{t}\left(\log\frac{t}{s}\right)^{-\alpha}\frac{\left|u\left(s\right)\right|}{s}ds\frac{dt}{t}.$$

As φ is nonincreasing, we have $\varphi\left(s\right)\geq\varphi\left(t\right)$ for all $t\geq s$ and $\frac{1}{\varphi^{1/m}(s)}\leq\frac{1}{\varphi^{1/m}(t)}$, $m>1$.

Also it is clear that

$$\varphi^{'}\left(t\right)=0,\quad t\in\left[a,\theta T\right].$$

Therefore

$$\int_{a}^{T}\delta\varphi\left(t\right)\left(\mathcal{J}_{a^{+}}^{1-\alpha}u\right)\left(t\right)\frac{dt}{t}$$

$$\leq\frac{1}{\Gamma\left(1-\alpha\right)}\int_{a}^{T}\left|\delta\varphi\left(t\right)\right|\int_{a}^{t}\left(\log\frac{t}{s}\right)^{-\alpha}\frac{\left|u\left(s\right)\right|\varphi^{1/m}\left(s\right)}{s\varphi^{1/m}\left(s\right)}ds\frac{dt}{t}$$

$$\leq\frac{1}{\Gamma\left(1-\alpha\right)}\int_{\theta T}^{T}\frac{\left|\delta\varphi\left(t\right)\right|}{\varphi^{1/m}\left(t\right)}\int_{a}^{t}\left(\log\frac{t}{s}\right)^{-\alpha}\frac{\left|u\left(s\right)\right|\varphi^{1/m}\left(s\right)}{s}ds\frac{dt}{t}.$$

Definition 2.3.1 allows us to write

$$\int_{a}^{T}\delta\varphi\left(t\right)\left(\mathcal{J}_{a^{+}}^{1-\alpha}u\right)\left(t\right)\frac{dt}{t}\leq\int_{\theta T}^{T}\frac{\left|\delta\varphi\left(t\right)\right|}{\varphi^{1/m}\left(t\right)}\left(\mathcal{J}_{a^{+}}^{1-\alpha}\left|u\right|\varphi^{1/m}\right)\left(t\right)\frac{dt}{t}.$$

By the same argument as the one used at the beginning of the proof we may show that $|u(t)|\varphi^{1/m}(t) \in X^{p'}_{-1/p}$ $(|u(t)|\varphi^{1/m}(t) < |u(t)|)$. Moreover, it is easy to see that $\frac{|\delta\varphi(t)|}{\varphi^{1/m}(t)} \in L_p$ (for otherwise we consider $\varphi^{\lambda}(t)$ with some sufficiently large λ). Thus, we can apply Lemma 2.3.6 to get

$$\int_a^T \delta\varphi(t)\left(\mathcal{J}_{a^+}^{1-\alpha}u\right)(t)\frac{dt}{t} \leq \int_{\theta T}^T |u(t)|\varphi^{1/m}(t)\left(\mathcal{J}_{T^-}^{1-\alpha}\frac{|\delta\varphi|}{\varphi^{1/m}}\right)(t)\frac{dt}{t}. \tag{4.56}$$

Next, we multiply by $\left(\log\frac{t}{a}\right)^{\mu/m}.\left(\log\frac{t}{a}\right)^{-\mu/m}$ inside the integral in the right hand side of (4.56)

$$\int_a^T \delta\varphi(t)\left(\mathcal{J}_{a^+}^{1-\alpha}u\right)(t)\frac{dt}{t} \leq \int_{\theta T}^T \left(\mathcal{J}_{T^-}^{1-\alpha}\frac{|\delta\varphi|}{\varphi^{1/m}}\right)(t)\,|u(t)|\varphi^{1/m}(t)\,\frac{\left(\log\frac{t}{a}\right)^{\mu/m}}{\left(\log\frac{t}{a}\right)^{\mu/m}}\frac{dt}{t}.$$

For $\mu \geq 0$ we have $\left(\log\frac{t}{a}\right)^{-\mu/m} \leq \left(\log\frac{\theta T}{a}\right)^{-\mu/m}$ (because $-\mu/m < 0$ and $t > \theta T$), that is

$$\left(\log\frac{t}{a}\right)^{-\mu/m} \leq \left(\log\frac{\theta T}{a}\right)^{-\mu/m}.$$

It follows that

$$\int_a^T \delta\varphi(t)\left(\mathcal{J}_{a^+}^{1-\alpha}u\right)(t)\frac{dt}{t}$$
$$\leq \left(\log\frac{\theta T}{a}\right)^{-\mu/m}\int_{\theta T}^T \left(\mathcal{J}_{T^-}^{1-\alpha}\frac{|\delta\varphi|}{\varphi^{1/m}}\right)(t)\left(\log\frac{t}{a}\right)^{\mu/m}|u(t)|\varphi^{1/m}(t)\frac{dt}{t}. \tag{4.57}$$

By using the Young inequality (see Theorem 2.5.2), with m and m' such that $\frac{1}{m}+\frac{1}{m'} = 1$, in the right hand side of (4.57) we find

$$\int_a^T \delta\varphi(t)\left(\mathcal{J}_{a^+}^{1-\alpha}u\right)(t)\frac{dt}{t}$$

$$\leq \frac{1}{m}\int_{\theta T}^{T}\left(\log\frac{t}{a}\right)^{\mu}\varphi(t)\left|u(t)\right|^{m}\frac{dt}{t}+\frac{\left(\log\frac{\theta T}{a}\right)^{-\mu m^{'}/m}}{m^{'}}\int_{\theta T}^{T}\left(\mathcal{J}_{T^{-}}^{1-\alpha}\frac{|\delta\varphi|}{\varphi^{1/m}}\right)^{m^{'}}(t)\frac{dt}{t}$$

$$\leq \frac{1}{m}\int_{a}^{T}\left(\log\frac{t}{a}\right)^{\mu}\varphi(t)\left|u(t)\right|^{m}\frac{dt}{t}+\frac{\left(\log\frac{\theta T}{a}\right)^{-\mu m^{'}/m}}{m^{'}}\int_{\theta T}^{T}\left(\mathcal{J}_{T^{-}}^{1-\alpha}\frac{|\delta\varphi|}{\varphi^{1/m}}\right)^{m^{'}}(t)\frac{dt}{t}. \tag{4.58}$$

Clearly from (4.55) and (4.58) we see that

$$-b\left(\mathcal{J}_{T^{-}}^{\beta(1-\alpha)}\varphi\right)\left(a^{+}\right)+\frac{\left(\log\frac{\theta T}{a}\right)^{-\mu m^{'}/m}}{m^{'}}\int_{\theta T}^{T}\left(\mathcal{J}_{T^{-}}^{1-\alpha}\frac{|\delta\varphi|}{\varphi^{1/m}}\right)^{m^{'}}(t)\frac{dt}{t}$$

$$\geq\left(1-\frac{1}{m}\right)\int_{a}^{T}\left(\log\frac{t}{a}\right)^{\mu}\varphi(t)\left|u(t)\right|^{m}\frac{dt}{t},$$

or, since $b\geq 0$ and $\varphi(t)\geq 0$ ($b\left(\mathcal{J}_{T^{-}}^{\beta(1-\alpha)}\varphi\right)\left(a^{+}\right)\geq 0$)

$$\frac{1}{m^{'}}\int_{a}^{T}\left(\log\frac{t}{a}\right)^{\mu}\varphi(t)\left|u(t)\right|^{m}\frac{dt}{t}\leq\frac{\left(\log\frac{\theta T}{a}\right)^{-\mu m^{'}/m}}{m^{'}}\int_{\theta T}^{T}\left(\mathcal{J}_{T^{-}}^{1-\alpha}\frac{|\delta\varphi|}{\varphi^{1/m}}\right)^{m^{'}}(t)\frac{dt}{t}.$$

Therefore, by Definition 2.3.2 we have

$$\int_{a}^{T}\left(\log\frac{t}{a}\right)^{\mu}\varphi(t)\left|u(t)\right|^{m}\frac{dt}{t}$$

$$\leq\frac{\left(\log\frac{\theta T}{a}\right)^{-\mu m^{'}/m}}{\Gamma^{m^{'}}(1-\alpha)}\int_{\theta T}^{T}\left(\int_{t}^{T}\left(\log\frac{s}{t}\right)^{-\alpha}\frac{|\delta\varphi(s)|}{\varphi^{1/m}(s)}\frac{ds}{s}\right)^{m^{'}}\frac{dt}{t}.$$

The change of variable $\sigma T=t$ yields

$$\int_{a}^{T}\left(\log\frac{t}{a}\right)^{\mu}\varphi(t)\left|u(t)\right|^{m}\frac{dt}{t}\leq\frac{\left(\log\frac{\theta T}{a}\right)^{-\frac{\mu m^{'}}{m}}}{\Gamma^{m^{'}}(1-\alpha)}\int_{\theta}^{1}\left(\int_{\sigma T}^{T}\left(\log\frac{s}{\sigma T}\right)^{-\alpha}\frac{\left|\varphi^{'}(s)\right|}{\varphi(s)^{1/m}}ds\right)^{m^{'}}\frac{d\sigma}{\sigma}.$$

Another change of variable $r = s/T$ gives

$$\int_a^T \left(\log \frac{t}{a}\right)^{\mu} \varphi(t) |u(t)|^m \frac{dt}{t}$$

$$\leq \frac{\left(\log \frac{\theta T}{a}\right)^{-\mu m'/m}}{\Gamma^{m'}(1-\alpha)} \int_\theta^1 \left(\int_\sigma^1 \left(\log \frac{r}{\sigma}\right)^{-\alpha} \frac{|\varphi'(r)|}{\varphi(r)^{1/m}} dr\right)^{m'} \frac{d\sigma}{\sigma}. \tag{4.59}$$

We may assume that the integral term in the right hand side of (4.59) is convergent, that is

$$\frac{1}{\Gamma^{m'}(1-\alpha)} \int_\theta^1 \left(\int_\sigma^1 \left(\ln \frac{r}{\sigma}\right)^{-\alpha} \frac{|\varphi'(r)|}{\varphi(r)^{1/m}} dr\right)^{m'} d\sigma \leq C$$

for some positive constant C, for otherwise we consider $\varphi^{\lambda}(r)$ with some sufficiently large λ. Therefore

$$\int_a^T \left(\log \frac{t}{a}\right)^{\mu} \varphi(t) |u(t)|^m \frac{dt}{t} \leq C \left(\log \frac{\theta T}{a}\right)^{-\mu m'/m} \tag{4.60}$$

If $\mu > 0$, we claim that $-\frac{\mu m'}{m} < 0$, and consequently

$$\left(\log \frac{\theta T}{a}\right)^{-\mu m'/m} \to 0$$

as $T \to \infty$. Finally from (4.60) we obtain

$$\lim_{T\to\infty} \int_a^T \left(\log \frac{t}{a}\right)^{\mu} \varphi(t) |u(t)|^m \frac{dt}{t} = 0.$$

We reach a contradiction since the solution is not supposed to be trivial.

In the case $\mu = 0$ we have $-\mu m^{'}/m = 0$ and the relation (4.60) ensures that

$$\lim_{T\to\infty} \int_a^T \left(\log \frac{t}{a}\right)^{\mu} \varphi(t) |u(t)|^m \frac{dt}{t} \le C. \tag{4.61}$$

Moreover, it is clear that

$$\left(\log \frac{\theta T}{a}\right)^{-\mu/m} \int_{\theta T}^{T} \left(\mathcal{J}_{T^-}^{1-\alpha} \frac{|\delta\varphi|}{\varphi^{1/m}}\right)(t) \left(\log \frac{t}{a}\right)^{\mu/m} |u(t)| \varphi^{1/m}(t) \frac{dt}{t}$$

$$\le \left(\log \frac{\theta T}{a}\right)^{-\mu/m} \left[\int_{\theta T}^{T} \left(\mathcal{J}_{T^-}^{1-\alpha} \frac{|\delta\varphi|}{\varphi^{1/m}}\right)^{m^{'}} (t) \frac{dt}{t}\right]^{\frac{1}{m^{'}}} \left[\int_{\theta T}^{T} \left(\log \frac{t}{a}\right)^{\mu} |u(t)|^m \varphi(t) \frac{dt}{t}\right]^{\frac{1}{m}}.$$

This relation, together with (4.55) and (4.57), implies that

$$\int_a^T \left(\log \frac{t}{a}\right)^{\mu} \varphi(t) |u(t)|^m \frac{dt}{t} \le K \left[\int_{\theta T}^{T} \left(\log \frac{t}{a}\right)^{\mu} |u(t)|^m \varphi(t) \frac{dt}{t}\right]^{\frac{1}{m}}$$

for some positive constant K, with

$$\lim_{T\to\infty} \int_{\theta T}^{T} \left(\log \frac{t}{a}\right)^{\mu} |u(t)|^m \varphi(t) \frac{dt}{t} = 0$$

due to the convergence of the integral in (4.61). This is again a contradiction. This completes the proof of Theorem 3.4.1.

Chapter 5

RECOMMENDATIONS

In this thesis we have restricted ourselves to the case $0 \leq \alpha, \beta \leq 1$. It will be certainly interesting to extend the Hilfer fractional derivative and Hilfer-Hadamard fractional derivative to the cases where α and β takes values above one.

It is also important to consider different types of nonlinearities in the stability and non-existence issues other than the ones we have investigated in this document.

Due to the time constraint the Hilfer fractional derivative has been defined and studied only for the Hadamard fractional derivative. It is worth introducing a similar Hilfer type fractional derivative which interpolates the Erdélyi-Kober fractional derivative and the corresponding Caputo type modification of the Erdélyi-Kober fractional derivative.

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